Math 6730 : Asymptotic and Perturbation Methods

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Preface

These notes are largely based on **Math 6730: Asymptotic and Perturbation Methods** course, taught by Paul Bressloff in Fall 2017, at the University of Utah. The main textbook is [Hol12], but additional examples or remarks or results from other sources are added as we see fit, mainly to facilitate our understanding. These notes are by no means accurate or applicable, and any mistakes here are of course our own. Please report any typographical errors or mathematical fallacy to us by email hkim@math.utah.edu or tan@math.utah.edu.

Chapter 1

Introduction to Asymptotic Approximation

Our main goal is to construct approximate solutions of differential equations to gain insight of the problem, since they are nearly impossible to solve analytically in general due to the nonlinear nature of the problem. Among the most important machinery in approximating functions in some small neighbourhood is the **Taylor's theorem**: Given $f \in C^{(N+1)}(B_\delta(x_0))$, for any $x \in B_\delta(x_0)$ we can write $f(x)$ as

$$f(x) = \sum_{k=0}^{N} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_{N+1}(x),$$

where $R_{N+1}(x)$ is the remainder term

$$R_{N+1}(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!}(x - x_0)^{N+1}$$

and ξ is a point between x and x_0. Taylor's theorem can be used to solve the following problem:

Given a certain tolerance $\varepsilon = |x - x_0| > 0$, how many terms should we include in the Taylor polynomial to achieve that accuracy?

Asymptotic approximation concerns about a slightly different problem:

Given a fixed number of terms N, how accurate is the asymptotic approximation as $\varepsilon \longrightarrow 0$?

We want to avoid from including as many terms as possible as $\varepsilon \longrightarrow 0$ and in contrast to Taylor's theorem, we do not care about convergence of the asymptotic approximation. In fact, most asymptotic approximations diverge as $N \longrightarrow \infty$ for a fixed ε.

Remark 1.0.1. If the given function is sufficiently differentiable, then Taylor's theorem offers a reasonable approximation and we can easily analyse the error as well.

1.1 Asymptotic Expansion

We begin the section with a motivating example. Suppose we want to evaluate the integral

$$f(\varepsilon) = \int_0^\infty \frac{e^{-t}}{1+\varepsilon t}\,dt, \quad \varepsilon > 0.$$

We can develop an approximation of $f(\varepsilon)$ for sufficiently small $\varepsilon > 0$ by repeatedly integrating by parts. Indeed,

$$\begin{aligned} f(\varepsilon) &= 1 - \varepsilon \int_0^\infty \frac{e^{-t}}{(1+\varepsilon t)^2}\,dt \\ &= 1 - \varepsilon + 2\varepsilon^2 - 6\varepsilon^3 + \cdots + (-1)^N N!\varepsilon^N + R_N(\varepsilon) \\ &= \sum_{k=0}^N a_k \varepsilon^k + R_N(\varepsilon), \end{aligned}$$

where

$$R_N(\varepsilon) = (-1)^{N+1}(N+1)!\varepsilon^{N+1} \int_0^\infty \frac{e^{-t}}{(1+\varepsilon t)^{N+2}}\,dt.$$

Since

$$\int_0^\infty \frac{e^{-t}}{(1+\varepsilon t)^{N+2}}\,dt \leq \int_0^\infty e^{-t}\,dt = 1,$$

it follows that

$$|R_N(\varepsilon)| \leq |(N+1)!\varepsilon^{N+1}|.$$

Thus, for fixed $N > 0$ we have that

$$\lim_{\varepsilon \to 0} \left| \frac{f(\varepsilon) - \sum_{k=0}^N a_k \varepsilon^k}{\varepsilon^N} \right| = 0$$

or

$$f(\varepsilon) = \sum_{k=0}^N a_k \varepsilon^k + o\left(\varepsilon^N\right) = \sum_{k=0}^N a_k \varepsilon^k + \mathcal{O}\left(\varepsilon^{N+1}\right).$$

The formal series $\displaystyle\sum_{k=0}^N a_k \varepsilon^k$ is said to be an asymptotic expansion of $f(\varepsilon)$ such that for fixed N, it provides a good approximation to $f(\varepsilon)$ as $\varepsilon \longrightarrow 0$. However, this expansion is not convergent for any fixed $\varepsilon > 0$, since

$$(-1)^N N!\varepsilon^N \longrightarrow \infty \quad \text{as } \varepsilon \longrightarrow 0,$$

i.e. the correction term actually blows up!

Remark 1.1.1. Observe that for sufficiently small $\varepsilon > 0$,

$$|R_N(\varepsilon)| \ll |(-1)^N N!\varepsilon^N|,$$

which means that the remainder $R_N(\varepsilon)$ is dominated by the $(N+1)$th term of the approximation, *i.e.* the error is of higher-order of the approximating function. This property is something that we would want to impose on the asymptotic expansion, and this idea can be made precise using the Landau symbols.

1.1.1 Order symbols

Definition 1.1.2.

1. $f(\varepsilon) = \mathcal{O}(g(\varepsilon))$ as $\varepsilon \longrightarrow 0$ means that there exists a finite M for which
$$|f(\varepsilon)| \leq M|g(\varepsilon)| \quad \text{as } \varepsilon \longrightarrow 0.$$

2. $f(\varepsilon) = o(g(\varepsilon))$ as $\varepsilon \longrightarrow 0$ means that
$$\lim_{\varepsilon\to 0}\left|\frac{f(\varepsilon)}{g(\varepsilon)}\right| = 0.$$

3. The ordered sequence of functions $\{\phi_k(\varepsilon)\}_{k=0}^{\infty}$ is called an **asymptotic sequence** as $\varepsilon \longrightarrow 0$ if and only if
$$\phi_{k+1}(\varepsilon) = o(\phi_k(\varepsilon)) \quad \text{as } \varepsilon \longrightarrow 0 \text{ for each } k.$$

4. Let $f(\varepsilon)$ be a continuous function of ε and $\{\phi_k(\varepsilon)\}_{k=0}^{\infty}$ an asymptotic sequence. The formal series expansion
$$\sum_{k=0}^{N} a_k\phi_k(\varepsilon)$$
is called an **asymptotic expansion valid to order $\boldsymbol{\phi_N(\varepsilon)}$** if for any $N \geq 0$,
$$\lim_{\varepsilon\to 0}\left|\frac{f(\varepsilon) - \sum_{k=0}^{N} a_k\phi_k(\varepsilon)}{\phi_N(\varepsilon)}\right| = 0.$$
We typically writes $f(\varepsilon) \sim \displaystyle\sum_{k=0}^{N} a_k\phi_k(\varepsilon)$ as $\varepsilon \longrightarrow 0$.

Remark 1.1.3. Intuitively, an asymptotic expansion of a given function f is a finite sum which might diverges, yet it still provides an increasingly accurate description of the asymptotic behaviour of f as $\varepsilon \longrightarrow 0$. There is a caveat here: for a divergent asymptotic expansion, for some ε, there exists an optimal $N_0 = N_0(\varepsilon)$ that gives best approximation to f, *i.e.* adding more terms actually gives worse accuracy. However, for values of ε sufficiently close to the limiting value 0, the optimal number of terms required increases, *i.e.* for every $\varepsilon_1 > 0$, there exists an δ and an optimal $N_0 = N_0(\delta)$ such that
$$\left|f(\varepsilon) - \sum_{k=0}^{N} a_k\phi_k(\varepsilon)\right| < \varepsilon_1 \quad \text{for every } |z - z_0| < \delta \text{ and } N > N_0.$$

Sometimes in approximating general solutions of ODEs, we will need to consider time-dependent asymptotic expansions. Suppose $\dot{x} = f(x,\varepsilon), x \in \mathbb{R}^n$. We seek a solution of the form
$$x(t,\varepsilon) \sim \sum_{k=0}^{N} a_k(t)\phi_k(\varepsilon) \quad \text{as } \varepsilon \longrightarrow 0,$$

which will tend to be valid over some range of times t. It is often useful to characterise the time interval over which the asymptotic expansion exists. We say that this estimate is valid on a time-scale $1/\hat{\delta}(\varepsilon)$ if

$$\lim_{\varepsilon \to 0} \left| \frac{x(t,\varepsilon) - \sum_{k=0}^{N} a_k(t)\phi_k(\varepsilon)}{\phi_N(\varepsilon)} \right| = 0 \quad \text{for } 0 \le \hat{\delta}(\varepsilon)t \le C,$$

for some C independent of ε.

1.1.2 Accuracy vs convergence

In the case of a Taylor series expansion, one can increase the accuracy (for fixed ε) by including more terms in the approximation, assuming we are expanding within the radius of convergence. This is not usually the case for an asymptotic expansion because the asymptotic expansion concerns the limit as $\varepsilon \longrightarrow 0$ whereas increasing the number of terms concerns $N \longrightarrow \infty$ for fixed ε.

1.1.3 Manipulating asymptotic expansions

Two asymptotic expansions can be added together term by term, assuming both involve the same basis functions $\{\phi_k(\varepsilon)\}$. Multiplication can also be carried out provided the asymptotic sequence $\{\phi_k(\varepsilon)\}$ can be ordered in a particular way. What about differentiation? Suppose

$$f(x,\varepsilon) \sim \phi_1(x,\varepsilon) + \phi_2(x,\varepsilon) \quad \text{as } \varepsilon \longrightarrow 0.$$

It is not necessarily the case that

$$\frac{d}{dx} f(x,\varepsilon) \sim \frac{d}{dx}\phi_1(x,\varepsilon) + \frac{d}{dx}\phi_2(x,\varepsilon) \quad \text{as } \varepsilon \longrightarrow 0.$$

There are two possible scenarios:

Example 1.1.4. Consider $f(x,\varepsilon) = e^{-x/\varepsilon}\sin\big(e^{x/\varepsilon}\big)$. Observe that for $x > 0$ we have that

$$\lim_{\varepsilon \to 0} \left| \frac{f(x,\varepsilon)}{\varepsilon^n} \right| = 0 \quad \text{for all finite } n,$$

which means that

$$f(x,\varepsilon) \sim 0 + 0 \cdot \varepsilon + 0 \cdot \varepsilon^2 + \dots \quad \text{as } \varepsilon \longrightarrow 0.$$

However,

$$\frac{d}{dx} f(x,\varepsilon) = -\frac{1}{\varepsilon} e^{-x/\varepsilon}\sin\big(e^{x/\varepsilon}\big) + \frac{1}{\varepsilon}\cos\big(e^{x/\varepsilon}\big) \longrightarrow \infty \quad \text{as } \varepsilon \longrightarrow 0.$$

i.e. the derivative cannot be expanded using the asymptotic sequence $\{1, \varepsilon, \varepsilon^2, \dots\}$.

Example 1.1.5. Even if $\{\phi_k(\varepsilon)\}$ is an ordered asymptotic sequence, its derivative $\{\phi_k'(\varepsilon)\}$ need not be. Consider $\phi_1(x) = 1 + x$, $\phi_2(x) = \varepsilon\sin(x/\varepsilon)$ for $x \in (0, 1)$. Then $\phi_2 = o(\phi_1)$ but

$$\phi_1'(x) = 1, \quad \phi_2'(x) = \cos(x/\varepsilon),$$

which are not ordered!

On the bright side, if

$$f(x,\varepsilon) \sim a_1(x)\phi_1(\varepsilon) + a_2(x)\phi_2(\varepsilon) \quad \text{as } \varepsilon \longrightarrow 0, \tag{1.1.1}$$

and if

$$\frac{d}{dx}f(x,\varepsilon) \sim b_1(x)\phi_1(\varepsilon) + b_2(x)\phi_2(\varepsilon) \quad \text{as } \varepsilon \longrightarrow 0, \tag{1.1.2}$$

then $b_k = \dfrac{da_k}{dx}$, *i.e.* the asymptotic expansion for $\dfrac{df}{dx}$ can be obtained from term by term differentiation of (1.1.1). Throughout this course, we will assume that (**??**) holds whenever we are given (1.1.1) which almost holds in practice. Integration, on the other hand, is less problematic. If

$$f(x,\varepsilon) \sim a_1(x)\phi_1(\varepsilon) + a_2(x)\phi_2(\varepsilon) \quad \text{as } \varepsilon \longrightarrow 0 \text{ for } x \in [a,b],$$

and all the functions are integrable, then

$$\int_a^b f(x,\varepsilon)\,dx \sim \left(\int_a^b a_1(x)\,dx\right)\phi_1(\varepsilon) + \left(\int_a^b a_2(x)\,dx\right)\phi_2(\varepsilon) \quad \text{as } \varepsilon \longrightarrow 0.$$

1.2 Algebraic and Transcendental Equations

We study three examples where approximate solutions are found using asymptotic expansions, but each uses different method. They serve to illustrate the important point that instead of performing the routine procedure with standard asymptotic sequence, we should taylor our asymptotic expansion to extract the physical property or behavior of our problem.

1.2.1 Singular quadratic equation

Consider the quadratic equation

$$\varepsilon x^2 + 2x - 1 = 0. \tag{1.2.1}$$

This is known as a **singular** problem since the order of the polynomial (and thus the nature of the equation) changes when $\varepsilon = 0$; in this case the unique solution is $x = 1/2$. It is evident from Figure 1.1 that there are two real roots for sufficiently small ε; one is located slightly to the left of $x = 1/2$ and one far left on the x-axis. This means that the asymptotic expansion should not start out as

$$x(\varepsilon) \sim \varepsilon x_0 + \dots,$$

because then $x(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Therefore, we try the asymptotic expansion

$$x(\varepsilon) \sim x_0 + \varepsilon^\alpha x_1 + \dots \quad \text{as } \varepsilon \longrightarrow 0, \tag{1.2.2}$$

for some $\alpha > 0$. Substituting (1.2.2) into (1.2.1) leads to

$$\underbrace{\varepsilon\left[x_0^2 + 2\varepsilon^\alpha x_0 x_1 + \dots\right]}_{①} + \underbrace{2\left[x_0 + \varepsilon^\alpha x_1 + \dots\right]}_{②} - 1 = 0. \tag{1.2.3}$$

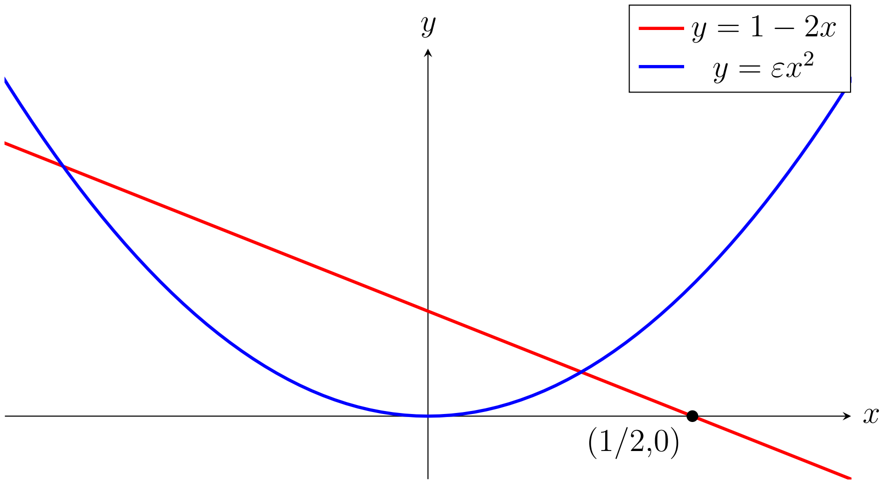


Figure 1.1: Graphs of $y = 1 - 2x$ and $y = \varepsilon x^2$.

Requiring (1.2.3) to hold as $\varepsilon \longrightarrow 0$ results in the $\mathcal{O}(1)$ equation

$$2x_0 - 1 = 0 \implies x_0 = \frac{1}{2}.$$

Since the right-hand side is zero, the $\mathcal{O}(\varepsilon)$ in ① must be balanced by the $\mathcal{O}(\varepsilon^\alpha)$ term in ②. This means that we must choose $\alpha = 1$ and the $\mathcal{O}(\varepsilon)$ equation is

$$x_0^2 + 2x_1 = 0 \implies x_1 = -\frac{1}{8}.$$

Consequently, a two-term expansion of one of the roots is

$$x^{(1)}(\varepsilon) \sim \frac{1}{2} - \frac{\varepsilon}{8} + \dots \quad \text{as } \varepsilon \longrightarrow 0.$$

The chosen ansatz (1.2.2) produce an approximation for the root near $x = 1/2$ and we missed the other root because it approaches negative infinity as $\varepsilon \longrightarrow 0$. One possible method to generate the other root is to consider solving

$$\varepsilon(x - x_1)(x - x_2) = 0,$$

but a more systematic method which is applicable to ODEs is to avoid the $O(1)$ solution. Take

$$x \sim \varepsilon^\gamma \left(x_0 + \varepsilon^\alpha x_1 + \dots\right) \quad \text{as } \varepsilon \longrightarrow 0, \tag{1.2.4}$$

for some $\alpha > 0$. Substituting (1.2.4) into (1.2.1) gives

$$\underbrace{\varepsilon^{(1+2\gamma)}\left[x_0^2 + 2\varepsilon^\alpha x_0 x_1 + \dots\right]}_{①} + \underbrace{2\varepsilon^\gamma\left[x_0 + \varepsilon^\alpha x_1 + \dots\right]}_{②} - \underbrace{1}_{③} = 0. \tag{1.2.5}$$

The terms on the LHS must balance to produce zero, and we need to determine the order of the problem that comes from this balancing. There are 3 possibilities on leading-order:

1. Set $\gamma = 0$ and we recover the root $x^{(1)}(\varepsilon)$ on balancing ② and ③.

2. Balance ① and ③, and so ② is higher-order. The condition ① $\sim$ ③ requires

$$1 + 2\gamma = 0 \implies \gamma = -\frac{1}{2},$$

so that the leading-order term in ①, ③ are of $\mathcal{O}(1)$, whilst ② $= \mathcal{O}(\varepsilon^{-1/2})$ which is lower order that ①. This is not possible.

3. Balance ① and ②, and so ③ is higher-order. The condition ① $\sim$ ② requires

$$1 + 2\gamma = \gamma \implies \gamma = -1,$$

so that the leading-order term in ①, ② are of $\mathcal{O}(\varepsilon^{-1})$ and ③ $= \mathcal{O}(1)$. This is consistent with the assumption!

Setting $\gamma = -1$ in (1.2.5) and multiplying by ε result in

$$\left(x_0^2 + 2\varepsilon^\alpha x_0 x_1 + \dots\dots\right) + 2\left(x_0 + \varepsilon^\alpha x_1 + \dots\dots\right) - \varepsilon = 0. \tag{1.2.6}$$

The $\mathcal{O}(1)$ equation is

$$x_0^2 + 2x_0 = 0 \implies x_0 = 0 \quad \text{or} \quad x_0 = -2.$$

The solution $x_0 = 0$ gives rise to the root $x^{(1)}(\varepsilon)$ by choosing $\alpha = 1$, so the new root is obtained by taking $x_0 = -2$. Balancing the equation as before means we must choose $\alpha = 1$ and the $\mathcal{O}(\varepsilon)$ equation is

$$2x_0x_1 + 2x_1 - 1 = 0 \implies x_1 = -\frac{1}{2}.$$

Hence, a two-term expansion of the second root of (1.2.2) is

$$x^{(2)}(\varepsilon) \sim \frac{1}{\varepsilon}\left(-2 - \frac{\varepsilon}{2}\right) \quad \text{as } \varepsilon \longrightarrow 0.$$

Remark 1.2.1. We may choose $x_0 = 1/2$ in (1.2.2) since one of the root should be close to $x = 1/2$ as we "switch on" ε in the term εx^2.

1.2.2 Exponential equation

Unlike algebraic equations, it is harder to determine the number of solutions of transcendental equations in most cases and we must resort to graphical method. Consider the equation

$$x^2 + e^{\varepsilon x} = 5 \tag{1.2.7}$$

From Figure 1.2, we see that there are two real solutions nearby $x = \pm 2$. We assume an asymptotic expansion of the form

$$x(\varepsilon) \sim x_0 + \varepsilon^{\alpha} x_1 + \dots \quad \text{as } \varepsilon \longrightarrow 0, \tag{1.2.8}$$

for some $\alpha > 0$. Substituting (1.2.8) into (1.2.7) and expanding the exponential term $e^{\varepsilon x}$ around $x = 0$ we obtain

$$\underbrace{\left[x_0^2 + 2\varepsilon^{\alpha} x_0 x_1 + \dots\right]}_{①} + \underbrace{\left[1 + \varepsilon x_0 + \dots\right]}_{②} = \underbrace{5}_{③}. \tag{1.2.9}$$

The $\mathcal{O}(1)$ equation is

$$x_0^2 + 1 = 5 \implies x_0 = \pm 2.$$

Balancing the $\mathcal{O}(\varepsilon^{\alpha})$ term in ① and the $\mathcal{O}(\varepsilon)$ term in ② gives $\alpha = 1$ and the $\mathcal{O}(\varepsilon)$ equation is

$$2x_0 x_1 + x_0 = 0 \implies x_1 = -\frac{1}{2}.$$

Hence, a two-term asymptotic expansion of each solution is

$$x(\varepsilon) \sim \pm 2 - \frac{\varepsilon}{2} \quad \text{as } \varepsilon \longrightarrow 0.$$

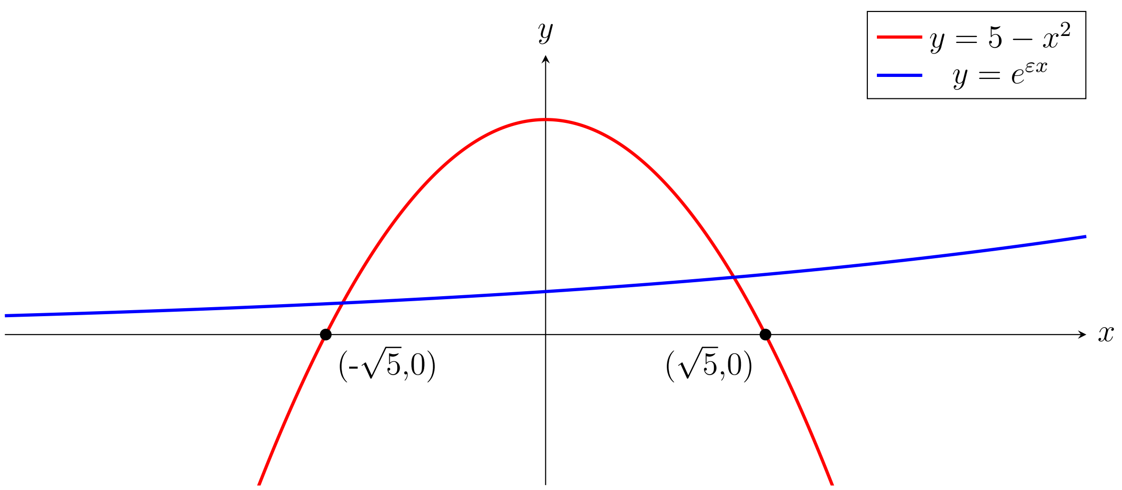


Figure 1.2: Graphs of $y = 5 - x^2$ and $y = e^{\varepsilon x}$.

1.2.3 Trigonometric equation

Consider the equation

$$x + 1 + \varepsilon \operatorname{sech}\left(\frac{x}{\varepsilon}\right) = 0. \tag{1.2.10}$$

It appears from Figure 1.3 that there exists a real solution and it approaches $x = -1$ as $\varepsilon \longrightarrow 0$. If we naively try

$$x \sim x_0 + \varepsilon^\alpha x_1 + \ldots \quad \text{as } \varepsilon \longrightarrow 0,$$

we obtain

$$[x_0 + \varepsilon^\alpha x_1 + \ldots] + 1 + \varepsilon \operatorname{sech}\left(\frac{x_0 + \varepsilon^\alpha x_1 + \ldots}{\varepsilon}\right) = 0$$

and it follows that $x_0 = -1$ since $\operatorname{sech}(x) \in (0, 1]$ for any $x \in \mathbb{R}$. However, we cannot balance subsequent leading-order terms since it is not possible to find α due to the nature of $\operatorname{sech}(x)$. From the definition of asymptotic sequences, we assume an asymptotic expansion of the form

$$x(\varepsilon) \sim x_0 + \mu(\varepsilon) \quad \text{as } \varepsilon \longrightarrow 0, \tag{1.2.11}$$

where we impose the condition $\mu(\varepsilon) \ll 1$ when $\varepsilon \ll 1$. Substituting (1.2.11) into (1.2.10) we obtain

$$[x_0 + \mu(\varepsilon)] + 1 + \varepsilon \operatorname{sech}\left[\frac{x_0}{\varepsilon} + \frac{\mu(\varepsilon)}{\varepsilon}\right] = 0. \tag{1.2.12}$$

The $\mathcal{O}(1)$ equation remains $x_0 = -1$ and (1.2.12) reduces to

$$\mu(\varepsilon) + \varepsilon \operatorname{sech}\left[\frac{x_0}{\varepsilon} + \frac{\mu(\varepsilon)}{\varepsilon}\right] = 0.$$

Since

$$\operatorname{sech}\left(\frac{x_0}{\varepsilon} + \frac{\mu(\varepsilon)}{\varepsilon}\right) \sim \operatorname{sech}\left[-\frac{1}{\varepsilon}\right] = \frac{2}{e^{1/\varepsilon} + e^{-1/\varepsilon}} \sim 2e^{-1/\varepsilon},$$

we require

$$\mu(\varepsilon) = -2\varepsilon e^{-1/\varepsilon} = o(1) \quad \text{as } \varepsilon \longrightarrow 0.$$

To construct the third term in the expansion, we would extend (1.2.11) into

$$x \sim -1 - 2\varepsilon e^{-1/\varepsilon} + \nu(\varepsilon),$$

where we impose the condition $\nu(\varepsilon) \ll \varepsilon e^{-1/\varepsilon}$.

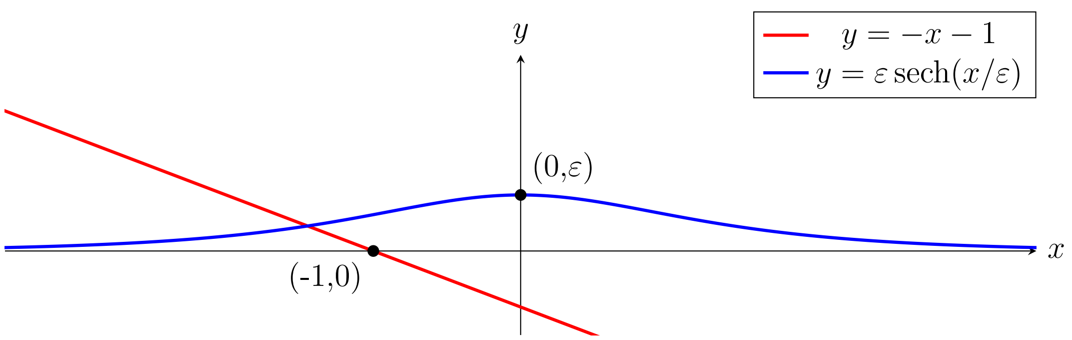


Figure 1.3: Graphs of $y = -x - 1$ and $y = \varepsilon \operatorname{sech}(x/\varepsilon)$.

1.3 Differential Equations: Regular Perturbation Theory

Roughly speaking, regular perturbation theory is a variant of Taylor's theorem, in the sense that we seek power series solution in ε. More precisely, we assume that the solution takes the form

$$x \sim x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots \quad \text{as } \varepsilon \longrightarrow 0,$$

where x_0 is the zeroth-order solution, *i.e.* the solution for the case $\varepsilon = 0$.

1.3.1 Projectile motion

Consider the motion of a gerbil projected radially upward from the surface of the Earth. Let $x(t)$ be the height of the gerbil from the surface of the Earth. Newton's law of motion asserts that

$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(x+R)^2}, \tag{1.3.1}$$

where R is the radius of the Earth and g is the gravitational constant. If $x \ll R$, then to a first approximation we obtain the initial value problem

$$\frac{d^2x}{dt^2} \approx -\frac{gR^2}{R^2} = -g, \quad x(0) = 0, \quad x'(0) = v_0, \tag{1.3.2}$$

where v_0 is some initial velocity. The solution is

$$x(t) = -\frac{gt^2}{2} + v_0 t. \tag{1.3.3}$$

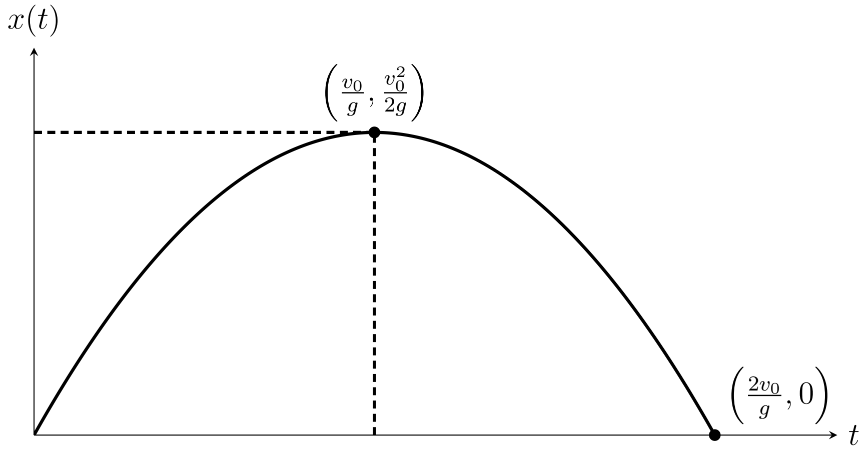


Figure 1.4: Graph of $x(t)$ versus t of the first approximation problem (1.3.2).

Unfortunately, this simplification does not determine a correction to the approximate solution (1.3.3). To this end, we nondimensionalise (1.3.1) with dimensionless variables

$$\tau = \frac{t}{t_c}, \quad y = \frac{x}{x_c},$$

where $t_c = v_0/g$ and $x_c = v_0^2/g$ are the chosen characteristic time and length scales respectively. This results in the dimensionless initial-value problem

$$\frac{d^2y}{d\tau^2} = -\frac{1}{(1+\varepsilon y)^2}, \quad y(0) = 0, \quad y'(0) = 1. \tag{1.3.4}$$

Observe that the dimensionless parameter $\varepsilon = \dfrac{x_c}{R} = \dfrac{v_0^2}{gR}$ measures how high the projectile gets in comparison to the radius of the Earth. Consider an asymptotic expansion

$$y(\tau) \sim y_0(\tau) + \varepsilon^\alpha y_1(\tau) + \dots \quad \text{as } \varepsilon \longrightarrow 0. \tag{1.3.5}$$

where the exponent $\alpha > 0$ is included since a-priori there is no reason to assume $\alpha = 1$. Assuming we can differentiate (1.3.5) term by term, we obtain using generalised Binomial theorem

$$\left[y_0'' + \varepsilon^\alpha y_1'' + \dots\right] = -\frac{1}{\left[1 + \varepsilon y_0 + \dots\right]^2} \sim -1 + 2\varepsilon y_0 + \dots,$$

with

$$y_0(0) + \varepsilon^\alpha y_1(0) + \dots = 0, \quad y_0'(0) + \varepsilon^\alpha y_1'(0) + \dots = 1.$$

The $\mathcal{O}(1)$ problem is

$$y_0'' = -1, \quad y_0(0) = 0, \quad y_0'(0) = 1 \implies y_0(\tau) = -\frac{\tau^2}{2} + \tau,$$

and we must choose $\alpha = 1$ to balance the term $2\varepsilon y_0$. Consequently, the $O(\varepsilon)$ problem is

$$y_1'' = 2y_0, \quad y_1(0) = 0, \quad y_1'(0) = 0 \implies y_1(\tau) = \frac{\tau^3}{3} - \frac{\tau^4}{12}.$$

Hence, a two-term asymptotic expansion of the solution of (1.3.4) is

$$y(\tau) \sim \tau\left(1 - \frac{1}{2}\tau\right) + \frac{1}{3}\varepsilon\tau^3\left(1 - \frac{\tau}{4}\right).$$

Note that the $\mathcal{O}(1)$ term is the scaled solution of (1.3.1) in a uniform gravitational field and the $\mathcal{O}(\varepsilon)$ term (first-order correction) contains the nonlinear effect of the problem.

1.3.2 Nonlinear potential problem

An interesting physical problem is the model of the diffusion of ions through a solution containing charged molecules. Assuming the solution occupies a domain Ω, the electrostatic potential $\phi(x)$ in the solution satisfies the **Poisson-Boltzmann equation**

$$\nabla^2\phi = -\sum_{i=1}^{k} \alpha_i z_i e^{-z_i\phi}, \quad x \in \Omega, \tag{1.3.6}$$

where the α_i are positive constants and z_i is the valence of the ith ionic species. The whole system must be neutral and this gives the electroneutrality condition

$$\sum_{i=1}^{k} \alpha_i z_i = 0. \tag{1.3.7}$$

We impose the Neumann boundary condition in which we assume the charge is uniform on the boundary:

$$\nabla\phi \cdot \boldsymbol{n} = \partial_{\boldsymbol{n}}\phi = \varepsilon \quad \text{on } \partial\Omega, \tag{1.3.8}$$

where $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega$.

This nonlinear problem has no known solutions. To deal with this, we invoke the classical **Debye-Hückle theory** in electrochemistry which assumes that the potential is small enough so that the Poisson-Boltzmann equation can be linearised. Because of the boundary condition (1.3.8), we may assume the zeroth-order solution is 0 and guess an asymptotic expansion of the form

$$\phi \sim \varepsilon\left(\phi_0(x) + \varepsilon\phi_1(x) + \dots\right) \quad \text{as } \varepsilon \longrightarrow 0, \tag{1.3.9}$$

where a small potential means ε is small. Substituting (1.3.9) into (1.3.6) and expanding the exponential function around the point 0 yields

$$\begin{aligned}
\varepsilon\left(\nabla^2\phi_0 + \varepsilon\nabla^2\phi + \dots\right) &= -\sum_{i=1}^k \alpha_i z_i e^{-\varepsilon z_i(\phi_0+\varepsilon\phi_1+\dots)} \\
&= -\sum_{i=1}^k \alpha_i z_i \left[1 - \varepsilon z_i\left(\phi_0 + \varepsilon\phi_1 + \dots\right) + \frac{1}{2}\varepsilon^2 z_i^2\left(\phi_0 + \varepsilon\phi_1 + \dots\right)^2 + \dots\right] \\
&= -\sum_{i=1}^k \alpha_i z_i \left[1 - \varepsilon z_i\phi_0 + \varepsilon^2\left(-z_i\phi_1 + \frac{1}{2}z_i^2\phi_0^2\right) + \dots\right] \\
&\sim \varepsilon\left(\sum_{i=1}^k \alpha_i z_i^2\phi_0\right) + \varepsilon^2\left(\sum_{i=1}^k \alpha_i z_i^2\left(\phi_1 - \frac{1}{2}z_i\phi_0^2\right)\right).
\end{aligned}$$

Setting $\kappa^2 = \sum_{i=1}^k \alpha_i z_i^2$, the $\mathcal{O}(\varepsilon)$ equation is

$$\nabla^2\phi_0 = \kappa^2\phi_0 \quad \text{in } \Omega, \tag{1.3.10a}$$
$$\partial_{\boldsymbol{n}}\phi_0 = 1 \quad \text{on } \partial\Omega. \tag{1.3.10b}$$

Setting $\lambda = \frac{1}{2}\sum_{i=1}^k \alpha_i z_i^3$, the $\mathcal{O}(\varepsilon^2)$ equation is

$$\left(\nabla^2 - \kappa^2\right)\phi_1 = -\lambda\phi_0^2 \quad \text{in } \Omega, \tag{1.3.11a}$$
$$\partial_{\boldsymbol{n}}\phi_1 = 0 \quad \text{on } \partial\Omega. \tag{1.3.11b}$$

Take Ω to be the region outside the unit sphere, which is radially symmetric. Writing the Laplacian operator ∇^2 in terms of spherical coordinates, the solution must be independent of the angular variables since the boundary condition is independent of the angular variables. With $\phi_0 = \phi_0(r)$, the $\mathcal{O}(\varepsilon)$ equation now has the form

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi_0}{dr}\right) - \kappa^2\phi_0 = 0 \quad \text{for } 1 < r < \infty, \tag{1.3.12a}$$
$$\phi_0'(1) = -1, \tag{1.3.12b}$$

where the negative sign is due to $\boldsymbol{n} = -\hat{\mathbf{r}}$. The bounded solution of (1.3.12) is

$$\phi_0(r) = \frac{1}{(1+\kappa)r} e^{\kappa(1-r)},$$

where the exponential term is the screening term. With $\phi_1 = \phi_1(r)$, the $\mathcal{O}(\varepsilon^2)$ equation takes the form

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi_0}{dr}\right) - \kappa^2\phi_1 = -\frac{\lambda}{(1+\kappa)^2 r^2} e^{2\kappa(1-r)} \quad \text{for } 1 < r < \infty, \tag{1.3.13a}$$

$$\phi_1'(1) = 0. \tag{1.3.13b}$$

Using *the method of variation of parameters*, the solution of (1.3.13) is

$$\begin{aligned}
\phi_1(r) &= \frac{\alpha}{r} e^{-\kappa r} + \frac{\gamma}{\kappa r}\left[e^{\kappa r} E_1(3\kappa r) - e^{-\kappa r} E_1(\kappa r)\right] \\
\gamma &= \frac{\lambda e^{2\kappa}}{2\kappa(1+\kappa)^2} \\
\alpha &= \frac{\gamma}{\kappa(1+\kappa)}\left[(\kappa-1)e^{2\kappa} E_1(3\kappa) + (\kappa+1)E_1(\kappa)\right] \\
E_1(z) &= \int_z^\infty \frac{e^{-t}}{t}\, dt.
\end{aligned}$$

1.3.3 Fredholm alternative

Let L_0 and L_1 be linear differential or integral operatots on the Hilbert space $L^2(\mathbb{R})$ with the standard inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)\, dx.$$

Consider the perturbed eigenvalue problem

$$(L_0 + \varepsilon L_1)\,\phi = \lambda\phi. \tag{1.3.14}$$

Spectral problems are widely studied in the context of time-dependence PDEs when time-harmonic solutions are sought for instance, and we are interested in the behaviour of the spectrum of L_0 as we perturb L_0. Suppose further that for $\varepsilon = 0$, the unperturbed equation has a unique solution (λ_0, ϕ_0) with λ_0 non-degenerate. For simplicity, take L_0 to be self-adjoint, that is

$$\langle f, L_0 g\rangle = \langle L_0 f, g\rangle.$$

Since L_0, L_1 are linear, we introduce the asymptotic expansions for both the eigenfunction ϕ and eigenvalue λ with asymptotic sequence $\{1, \varepsilon, \varepsilon^2, \dots\}$

$$\begin{aligned}
\phi &\sim \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \dots \\
\lambda &\sim \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \dots.
\end{aligned}$$

We obtain

$$(L_0 + \varepsilon L_1)\left[\phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \dots\right] = \left[\lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \dots\right]\left[\phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \dots\right].$$

The $\mathcal{O}(1)$ equation is $L_0\phi_0 = \lambda_0\phi_0$ and the $\mathcal{O}(\varepsilon)$ equation is

$$L_0\phi_1 + L_1\phi_0 = \lambda_0\phi_1 + \lambda_1\phi_0$$
$$(L_0 - \lambda_0 I)\,\phi_1 = \lambda_1\phi_0 - L_1\phi_0.$$

It follows from the Fredholm alternative that a necessary condition for the existence of $\phi_1 \in L^2(\mathbb{R})$ is that

$$(\lambda_1\phi_0 - L_1\phi_0) \in \ker((L_0 - \lambda_0 I)^*)^{\perp} = \ker(L_0 - \lambda_0 I)^{\perp},$$

and this in turn provides the solvability condition for λ_1. Since $\ker(L_0 - \lambda_0 I) = \text{span}(\phi_0)$ and L_0 is self-adjoint,

$$0 = \langle \phi_0, (L_0 - \lambda_0 I)\,\phi_1 \rangle = \lambda_1 \langle \phi_0, \phi_0 \rangle - \langle \phi_0, L_1\phi_0 \rangle$$
$$\lambda_1 = \frac{\langle \phi_0, L_1\phi_0 \rangle}{\langle \phi_0, \phi_0 \rangle}.$$

This expression of λ_1 represents the first-order correction to the eigenvalue of the operator $(L_0 + \varepsilon L_1)$. The $O(\varepsilon^n)$ equation can be analysed in a similar manner, where λ_n can be found using the solvability condition from the Fredholm alternative, assuming $\{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\}$ are non-degenerate.

1.4 Problems

1. Consider the transcendental equation

$$1 + \sqrt{x^2 + \varepsilon} = e^x. \tag{1.4.1}$$

Explain why there is only one small root for small ε. Find the three term expansion of the root

$$x \sim x_0 + x_1\varepsilon^{\alpha} + x_2\varepsilon^{\beta}, \quad \beta > \alpha > 0.$$

Solution: Consider two graph $f(x) = \sqrt{x^2 + \varepsilon}$ and $g(x) = e^x - 1$. If $x < 0$, then $f(x) > 0 > g(x)$. It means that there is no solution in negative region. If $x > 0$, then $f(x) \to x$ as $x \to \infty$ starting its curve from $f(0) = \varepsilon$. One can draw graph of $f(x)$ and $g(x)$ on $x > 0$, then it yields there is only one solution.

To obtain first expansion, set $\varepsilon = 0$. Then we get

$$1 + x = e^x \quad \Longrightarrow \quad x = 0.$$

Since there is only on solution for all $\varepsilon > 0$, then one can set $x_0 = 0$ and expand x further at this point. To do so, rewrite the equation (1.4.1) as

$$x^2 + \varepsilon = (e^x - 1)^2$$

and $x \ll 1$ as $\varepsilon \ll 1$, it is reasonable to expand RHS as Taylor series. Then one can obtain

$$x^2 + \varepsilon = \left(x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right)^2 = x^2 + x^3 + \frac{7}{12}x^4 + \cdots. \tag{1.4.2}$$

Before we balance both sides, consider the leading-order of both sides. Without doubt, the leading-order is $\varepsilon^{2\alpha}$ with coefficient x_1^2. For LHS, we have three cases

(a) If $2\alpha > 1$, then the leading-order is ε with coefficient 1. It leads to a contradiction when balancing both sides because $2\alpha = 1$. $(\times)$

(b) If $2\alpha = 1$, the balancing equation yields $x_1^2 + 1 = x_1^2$ and it does not make sense. $(\times)$

(c) Thus, the only case is $2\alpha < 1$.

Then one can rewrite equation (1.4.2) as

$$\varepsilon = x^3 + \frac{7}{12}x^4 + \cdots = (x_1^3\varepsilon^{3\alpha} + 3x_1^2x_2\varepsilon^{2\alpha+\beta} + \cdots) + \frac{7}{12}(x_1^4\varepsilon^{4\alpha} + \cdots).$$

Since the leading-order of RHS is $\varepsilon^{3\alpha}$, it provides that $1 = 3\alpha$ and $1 = x_1^3$. Thus, $\alpha = 1/3$ and $x_1 = 1$. The next leading term is $\varepsilon^{4\alpha}$. Since there is no remaining term on LHS, then balance RHS as

$$2\alpha + \beta = 4\alpha \quad \text{and} \quad 0 = 3x_1^2x_2 + \frac{7}{12}x_1^4,$$

yields $\beta = 2\alpha = 2/3$ and $x_2 = -7/36$. Therefore, the three term expansion of root is

$$x \sim 0 + \varepsilon^{1/3} - \frac{7}{36}\varepsilon^{2/3}. \tag{1.4.3}$$

2. A classical eigenvalue problem is the transcendental equation

$$\lambda = \tan(\lambda).$$

(a) After sketching the two functions in the equation, establish that there is an infinite number of solutions, and for sufficiently large λ takes the form

$$\lambda = \pi n + \frac{\pi}{2} - x_n,$$

with x_n small.

Solution: Tangent is π-periodic function with asymptotic line $\lambda_n = \pi n + \pi/2$. $\tan(\lambda) \to \infty$ as $\lambda \to \lambda_n^+$. Since $f(\lambda) = \lambda$ passes through all the asymptotic line and tangent function is close to the asymptotic line, then for sufficiently large n, λ takes the form

$$\lambda = \lambda_n - x_n$$

where x_n is a small number and tends to zero as $n \to \infty$.

(b) Find an asymptotic expansion of the large solutions of the form

$$\lambda \sim \varepsilon^{-\alpha}\left(\lambda_0 + \varepsilon^{\beta}\lambda_1\right),$$

and determine $\varepsilon, \alpha, \beta, \lambda_0, \lambda_1$.

Solution: Set $\lambda = 1/\varepsilon$ and see the asymptotic behavior of x_n. Then one can figure out an asymptotic expansion of λ. For convenience, set $x_n = x$. Then one can get

$$\frac{1}{\varepsilon} - x = \tan\left(\frac{1}{\varepsilon} - x\right) = \cot(x)$$

because $\tan(1/\varepsilon) = 0$. By multiplying $\varepsilon \tan(x)$ on both sides and we get

$$\tan(x) - \varepsilon x \tan(x) = \varepsilon.$$

Sicne we know that $x \to 0$ as $\varepsilon \to 0$, take $x \sim x_0 \varepsilon^\theta$, $\theta > 0$. It follows that

$$(x_0\varepsilon^\theta + \cdots) - \varepsilon(x_0\varepsilon^\theta + \cdots)(x_0\varepsilon^\theta + \cdots) = \varepsilon$$

Since $\theta > 0$, the leading-order of LHS is ε^θ. To balance both sides with $\mathcal{O}(\varepsilon)$, set $\theta = 1$ and get $x_0 = 1$. Therefore, an asymptotic expansion of λ is

$$\lambda = \frac{1}{\varepsilon} - x \sim \frac{1}{\varepsilon} - \varepsilon = \varepsilon^{-1}(1 + \varepsilon^2(-1)).$$

It follows that $\alpha = -1$, $\beta = 2$, $\lambda_0 = 1$ and $\lambda_1 = -1$.

3. In the study of porous media one is interested in determining the permeability $k(s) = F'(c(s))$, where

$$\int_0^1 F^{-1}(c - \varepsilon r)\, dr = s$$
$$F^{-1}(c) - F^{-1}(c - \varepsilon) = \beta,$$

and β is a given positive constant. The functions $F(c)$ and c both depend on ε, whereas s and β are independent of ε. Find the first term in the expansion of the permeability for small ε. *Hint: consider an asymptotic expansion of c and use the fact that s is independent of ε.*

Solution: Take $c \sim c_0 + c_1\varepsilon + \cdots$. Substituting it into given integral equation and expanding F^{-1} as Taylor series centered at $c = c_0$ yields

$$\int_0^1 F^{-1}(c_0) + \varepsilon(c_1 - r)\frac{dF^{-1}}{dc}(c_0) + \mathcal{O}(\varepsilon^2) dr$$
$$= F^{-1}(c_0) + \varepsilon\left(c_1 - \frac{1}{2}\right)\frac{dF^{-1}}{dc}(c_0) + \mathcal{O}(\varepsilon^2) = s.$$

Since s is independent of ε, then it gives us that

$$F^{-1}(c_0) = s \quad \text{and} \quad c_0 - \frac{1}{2} = 0 \implies c_0 = F(s) \quad \text{and } c_1 = \frac{1}{2}.$$

From the second condition, expanding F^{-1} as Taylor series follows provides that

$$F^{-1}(c_0) + \varepsilon c_1 \frac{dF^{-1}}{dc}(c_0) - F^{-1}(c_0) - \varepsilon(c_1 - 1)\frac{dF^{-1}}{dc}(c_0) + \mathcal{O}(\varepsilon^2) = \beta.$$

It follows that

$$\varepsilon\frac{1}{F'(s)} + \mathcal{O}(\varepsilon^2) = \beta \implies k(s) = F'(s) \sim \frac{\varepsilon}{\beta}.$$

4. Let A and D be real $n \times n$ matrices.

(a) Suppose A is symmetric and has n distinct eigenvalues. Find a two-term expansion of the eigenvalues of the perturbed matrix $A + \varepsilon D$, where D is positive definite.

Solution: We assume the asymptotic expansions of the eigenpairs (λ, x):

$$\lambda \sim \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \dots$$
$$x \sim x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots.$$

Substituting these into the eigenvalue equation $(A + \varepsilon D)x = \lambda x$ yields

$$(A + \varepsilon D)(x_0 + \varepsilon x_1 + \dots) = (\lambda_0 + \varepsilon\lambda_1 + \dots)(x_0 + \varepsilon x_1 + \dots).$$

The $\mathcal{O}(1)$ equation is $Ax_0 = \lambda_0 x_0$ which means that (λ_0, x_0) is the eigenpair of the matrix A. The $\mathcal{O}(\varepsilon)$ equation is

$$Ax_1 + Dx_0 = \lambda_0 x_1 + \lambda_1 x_0,$$

or

$$Lx_1 = (A - \lambda_0 I)x_1 = \lambda_1 x_0 - Dx_0.$$

It follows from the Fredholm Alternative that the solvability condition for λ_1 is

$$\lambda_1 \in \ker(L^T)^\perp = \ker(L)^\perp = \text{span}(x_0).$$

Consequently,

$$0 = x_0^T L x_1 = x_0^T(\lambda_1 x_0 - Dx_0) \implies \lambda_1 = \frac{x_0^T D x_0}{x_0^T x_0}.$$

(b) Consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Use this example to show that the $\mathcal{O}(\varepsilon)$ perturbation of a matrix need not result in a $\mathcal{O}(\varepsilon)$ perturbation of the eigenvalues, nor that the perturbation is smooth (at $\varepsilon = 0$).

Solution: The perturbed matrix $A + \varepsilon D = \begin{bmatrix} 0 & 1 \\ \varepsilon & 0 \end{bmatrix}$ has eigenvalues $\lambda = \pm\sqrt{\varepsilon}$, which is not of $\mathcal{O}(\varepsilon)$ and is not differentiable at $\varepsilon = 0$.

5. The eigenvalue problem for the vertical displacement $y(x)$ of an elastic string with variable density is
$$y'' + \lambda^2 \rho(x, \varepsilon) y = 0, \quad 0 < x < 1,$$
where $y(0) = y(1) = 0$. For small ε, assume $\rho \sim 1 + \varepsilon\mu(x)$, where $\mu(x)$ is positive and continuous. Consider the asymptotic expansions
$$y \sim y_0(x) + \varepsilon y_1(x), \quad \lambda \sim \lambda_0 + \varepsilon\lambda_1.$$

(a) Find y_0, λ_0 and λ_1. (The latter will involve an integral expression.)

Solution: Substituting the given asymptotic expansions together with the approximation $\rho \sim 1 + \varepsilon\mu(x)$ gives
$$[y_0'' + \varepsilon y_1'' + \dots] + [\lambda_0 + \varepsilon\lambda_1 + \dots]^2 [1 + \varepsilon\mu(x)] [y_0 + \varepsilon y_1 + \dots] = 0.$$
The $\mathcal{O}(1)$ equation is
$$y_0'' + \lambda^2 y_0 = 0, \quad y_0(0) = y_0(1) = 0,$$
and this boundary value problem has solutions
$$y_{0,n}(x) = A\sin(\lambda_{0,n}x) = A\sin(n\pi x), \quad n \in \mathbb{Z}.$$
The $\mathcal{O}(\varepsilon)$ equation is
$$y_1'' + \lambda_0^2 y_1 + \lambda_0^2\mu(x)y_0 + 2\lambda_0\lambda_1 y_0 = 0, \quad y_1(0) = y_1(1) = 0.$$
Using integration by parts, one can show that the linear operator $L = \frac{d^2}{dx^2} + \lambda_0^2$ with domain
$$\mathcal{D}(L) = \left\{ f \in C^2[0,1] \colon f(0) = f(1) = 0 \right\},$$
is self-adjoint with respect to the L^2 inner product over $[0,1]$. Moreover, for a fixed λ_0 it has a one-dimensional kernel $\ker(L) = \text{span}(\sin(\lambda_0 x))$. We can now determine λ_1 using Fredholm alternative, this results in
$$\begin{aligned} 0 &= \langle y_0, \lambda^2\mu(x)y_0\rangle + \langle y_0, 2\lambda_0\lambda_1 y_0\rangle \\ \lambda_1 &= -\frac{\lambda_0^2\langle y_0, \mu(x)y_0\rangle}{2\lambda_0\langle y_0, y_0\rangle} \\ &= -\lambda_0 \int_0^1 \mu(x)\sin^2(n\pi x)\,dx, \end{aligned}$$

since

$$\langle y_0, y_0 \rangle = \int_0^1 A^2 \sin^2(n\pi x)\, dx = A^2 \int_0^1 \frac{1-\cos(2n\pi x)}{2}\, dx = \frac{A^2}{2}.$$

(b) Using the equation for y_1, explain why the asymptotic expansion can break down when λ_0 is large.

Solution: From the previous results, one can find the equation for y_1 as

$$y_1''(x) + \lambda_0^2 y_1(x) = \lambda_0^2 \left(-\mu(x) y_0(x) + 2\int_0^1 \mu(s) y_0^2(s) ds \right).$$

Notice that the RHS proportional to λ_0^2 and it follows that the particular solution of y_1 is proportional to λ_0^2, then it implies that $y_1 \to \infty$. This can break down the expansion mixed with ε.

6. Consider the following eigenvalue problem:

$$\int_0^a K(x,s) y(s)\, ds = \lambda y(x), \quad 0 < x < a.$$

This is a Fredholm integral equation, where the kernel $K(x,d)$ is known and is assumed to be smooth and positive. The eigenfunction $y(x)$ is taken to be positive and normalized so that

$$\int_0^a y^2(s)\, ds = a.$$

Both $y(x)$ and λ depend on the parameter a, which is assumed to be small.

(a) Find the first two terms in the expansion of λ and $y(x)$ for small a.

Solution: Since the LHS of Fredholm integral equation is proportional to a, because integral contains a, the leading-order of eigenvalue λ is $\mathcal{O}(a)$. So, take

$$\lambda \sim \lambda_0 a + \lambda_1 a^2 \quad \text{ and } y(x) \sim y_0(x) + y_1(x) a.$$

Expand $K(x,s)$ and $y(s)$ as Taylor series in terms of s centered at $s=0$ because $0 < s < a$ is also small. Then we get

$$\int_0^a (K(x,0) + K_d(x,0)s + \cdots)(y(0) + y'(0)s + \cdots) ds = \lambda y(x),$$

and it follows that

$$aK(x,0)y(0) + \frac{a^2}{2}(K(x,0)y'(0) + K_d(x,0)y(0)) + \cdots = \lambda y(x).$$

Balance $\mathcal{O}(a)$ terms and one can obtain

$$K(x,0) y_0(0) = \lambda_0 y_0(x). \tag{1.4.4}$$

Balance $\mathcal{O}(a^2)$ terms and one can find

$$K(x,0)y_1(0) + \frac{1}{2}(K(x,0)y_0'(0) + K_d(x,0)y_0(0)) = \lambda_1 y_0(x) + \lambda_0 y_1(x). \quad (1.4.5)$$

In the same fashion, find one more asymptotic equation from given normalization equation

$$\int_0^a (y(0) + y'(0)s + \cdots)^2 ds = a$$

and it follows that

$$a \cdot (y(0))^2 + \frac{a^2}{2} \cdot 2y(0)y'(0) + \cdots = a.$$

Balance $\mathcal{O}(a)$ terms and one can obtain

$$(y_0(0))^2 = 1. \quad (1.4.6)$$

Balance $\mathcal{O}(a^2)$ terms and one can find

$$2y_0(0)y_1(0) + \frac{1}{2} \cdot 2y_0(0)y_0'(0) = 0. \quad (1.4.7)$$

From equation (1.4.4,1.4.6), one can find

$$y_0(0) = 1 \quad \text{and} \quad \lambda_0 = K(0,0).$$

This implies that

$$y_0(x) = \frac{K(x,0)}{K(0,0)}.$$

Similarly, one can find

$$\lambda_1 = \frac{1}{2}(K(0,0)y_0'(0) + K_d(0,0)) = \frac{1}{2}(K_x(0,0) + K_d(0,0))$$

and

$$y_1(x) = \frac{1}{\lambda_0}\left[K(x,0)y_1(0) + \frac{1}{2}\left(\frac{K(x,0)K_x(x,0)}{K(0,0)} + K_d(x,0)\right) - \lambda_1 y_0(x)\right],$$

and it follow that

$$y_1(x) = \frac{1}{2\lambda_0}\left[-\frac{K(x,0)}{K(0,0)}[K_x(x,0) - K_x(0,0)] + K_d(x,0) - 2\lambda_1 y_0(x)\right].$$

(b) By changing variables, transform the integral equation into

$$\int_0^1 K(a\xi, ar)\phi(r)\,dr = \frac{\lambda}{a}\phi(\xi), \quad 0 < \xi < 1.$$

Write down the normalisation condition for ϕ.

Solution: Substituting $x = a\xi$ and $s = ar$ into the Fredholm integral equation yields

$$\int_0^1 K(a\xi, ar)y(ar)adr = \lambda y(ar),$$

and set $\phi(r) = y(ar)$. It follows that

$$\int_0^1 K(a\xi, ar)\phi(r)dr = \frac{\lambda}{a}\phi(r).$$

In the same fashion, consider the normalization equation

$$\int_0^a y^2(s)ds = a \implies \int_0^1 \phi^2(r)adr = a \implies \int_0^1 \phi^2(r)dr = 1.$$

(c) From part (b) find the two-term expansion for λ and $\phi(\xi)$ for small a.

Solution: Take $\lambda \sim a\lambda_0 + a^2\lambda_1$ and $\phi \sim \phi_0 + a\phi_1$. In the same fashion we did in part (a), expand K inside of integral centered at zero

$$\begin{aligned}
&\int_0^1 (K(0,0) + K_x(0,0)a\xi + K_d(0,0)ar + \cdots)\phi(r)dr = \\
&\quad K(0,0)\int_0^1 \phi(r)dr + a\xi K_x(0,0)\int_0^1 \phi(r)dr + \\
&\quad aK_d(0,0)\int_0^1 r\phi(r)dr + \cdots.
\end{aligned}$$

Then balance $\mathcal{O}(1)$ terms in both sides of the equations and we get

$$\begin{cases} K(0,0)\int_0^1 \phi_0(r)dr = \lambda_0\phi_0(\xi) \\ \int_0^1 (\phi_0(r))^2 dr = 1 \end{cases}.$$

It implies that ϕ_0 is constant, and it yields that

$$\phi_0(\xi) = 1 \quad \text{and} \quad \lambda_0 = K(0,0). \tag{1.4.8}$$

Similarly, balance $\mathcal{O}(a)$ terms of the equations and one can obtain

$$\begin{cases} K(0,0)\int_0^1 \phi_1(r)dr + \xi K_x(0,0)\int_0^1 \phi_0(r)dr + K_d(0,0)\int_0^1 r\phi_0(r)dr \\ \qquad\qquad = \lambda_0\phi_1(\xi) + \lambda_1\phi_0(\xi) \\ \int_0^1 \phi_0(r)\phi_1(r)dr = 0 \end{cases}$$

It follows that $\int_0^1 \phi_1(r)dr = 0$ and one can have

$$\xi K_x(0,0) + \frac{1}{2}K_d(0,0) = \lambda_0\phi_1(\xi) + \lambda_1.$$

Integrate both side with respect to ξ on $[0,1]$ and get

$$\lambda_1 = \frac{1}{2}K_x(0,0) + \frac{1}{2}K_d(0,0). \tag{1.4.9}$$

This eigenvalue yields that

$$\phi_1(\xi) = \frac{K_x(0,0)}{\lambda_0}\left(\xi - \frac{1}{2}\right). \tag{1.4.10}$$

(d) Explain why the expansions in parts (a) and (c) are the same for λ but not the eigenfunction.

Solution: The eigenvalue is coordinate invariant, so it is not affected by change of variables. However, the eigenfunctions are.

7. In quantum mechanics, the perturbation theory for bound states involves the time-independent Schrodinger equation

$$\psi'' - [V_0(x) + \varepsilon V_1(x)]\,\psi = -E\psi, \quad -\infty < x < \infty,$$

where $\psi(-\infty) = \psi(\infty) = 0$. In this problem, the eigenvalue E represents energy and V_1 is a perturbing potential. Assume that the unperturbed ($\varepsilon = 0$) eigenvalue is nonzero and nondegenerate.

(a) Assuming

$$\psi(x) \sim \psi_0(x) + \varepsilon\psi_1(x) + \varepsilon^2\psi_2(x), \quad E \sim E_0 + \varepsilon E_1 + \varepsilon^2 E_2,$$

write down the equation for $\psi_0(x)$ and E_0. We will assume in the following that

$$\int_{-\infty}^{\infty} \psi_0^2(x)\,dx = 1, \quad \int_{-\infty}^{\infty} |V_1(x)|\,dx < \infty.$$

Solution: Substituting expansion of ψ and E into the Schrodinger equation, one can balance $\mathcal{O}(1)$ terms and get

$$\psi_0''(x) - V_0(x)\psi_0(x) = -E\psi_0(x).$$

(b) Substituting $\psi(x) = e^{\phi(x)}$ into the Schrodinger equation and derive the equation for $\phi(x)$.

Solution: Take derivative twice to ψ and it yields that

$$\psi'(x) = \phi'(x)e^{\phi(x)} \quad \text{and} \quad \psi''(x) = (\phi''(x) + (\phi'(x))^2)e^{\phi(x)}.$$

Plug them into the Schrodinger equation and drop common term $e^{\phi(x)}$. Then it follows that

$$\phi''(x) + (\phi'(x))^2 - (V_0(x) + \varepsilon V_1(x)) = -E.$$

(c) By expanding $\phi(x)$ for small ε, determine E_1 and E_2 in terms of ψ_0 and V_1.

Solution: Assume that $\phi(x) \sim \phi_0(x) + \varepsilon\phi_1(x) + \varepsilon^2\phi_2(x)$. Substituting it into the new Schrodinger equation and balance $\mathcal{O}(\varepsilon)$ terms. Then one can obtain

$$\phi_1'' + 2\phi_0'\phi_1' = V_1 - E_1.$$

Define an differential operator $L = d^2/dx^2 + 2\phi_0' \cdot d/dx$. Notice that for sufficiently smooth f,

$$\langle \psi_0^2, Lf \rangle = \int_{-\infty}^{\infty} e^{2\phi_0(x)}(f''(x) + 2\phi_0'(x)f'(x))dx.$$

Performing integration by parts

$$\langle \psi_0^2, Lf \rangle = \int_{-\infty}^{\infty} e^{2\phi_0(x)} f''(x)dx + e^{2\phi_0(x)} f'(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{2\phi_0(x)} f''(x)dx,$$

and it follows that $\langle \psi_0^2, Lf \rangle = 0$. From the observation, take inner product with ψ_0^2 to the first order balance equation and get

$$\langle \psi_0^2, L\phi_1 \rangle = 0 = \langle \psi_0^2, V_1 \rangle - E_1 \langle \psi_0^2, 1 \rangle = \langle \psi_0^2, V_1 \rangle - E_1.$$

Therefore,

$$E_1 = \langle \psi_0^2, V_1 \rangle = \int_{-\infty}^{\infty} V_1(x)[\psi_0(x)]^2 dx. \tag{1.4.11}$$

To find ϕ_1, solve the first order inhomogeneous ODE of ϕ_1' by integrating factor[a], or observe that

$$\int_{-\infty}^{x} \psi_0^2 L\phi_1 dy = \int_{-\infty}^{x} \frac{d}{dy}\left(\psi_0^2 \frac{d\phi_1}{dy}\right) dy = \psi_0^2(x)\phi_1'(x)$$
$$= \int_{-\infty}^{x} \psi_0^2 (V_1 - E_1) dy$$

and it yields that

$$\phi_1'(x) = \frac{1}{\psi_0^2(x)} \int_{-\infty}^{x} \psi_0^2(V_1 - E_1)dy.$$

Similarly, find the second order balance equation

$$\phi_2'' + 2\phi_0'\phi_2' + (\phi_1')^2 = -E_2 \implies L\phi_2 = -(\phi_1')^2 - E_2.$$

Therefore, $E_2 = -\langle \psi_0^2, (\phi_1')^2 \rangle$.

[a] In hierarchical system, use Green function, set Ansatz, or various DE methods.

Chapter 2

Matched Asymptotic Expansions

For most of singular perturbation problem of differential equations, the solution has extreme changes because a singular problem converges to a differential equation with different order or behavior as $\epsilon \to 0$. If we apply the regular asymptotic expansion, it fails to represent such drastic change and to match all boundary condition. To resolve the problem, we introduce matched asymptotic expansion which approximates the exact solution by zooming in the extreme changing zones, such as inner or boundary layers, together with the regular expansion for outer region.

2.1 Introductory example

Consider a singular problem

$$\begin{cases} \epsilon y'' + 2y' + 2y = 0 & ,0 < x < 1 \\ y(0) = y(1) = 1 \end{cases} \tag{2.1.1}$$

If $\epsilon = 0$, then we have a first order ODE. It only needs one boundary condition. It yields to have drastic dynamics on boundary layer. Remark that boundary layer could be interior, not only near boundary of domain.

2.1.1 Outer solution by regular perturbation

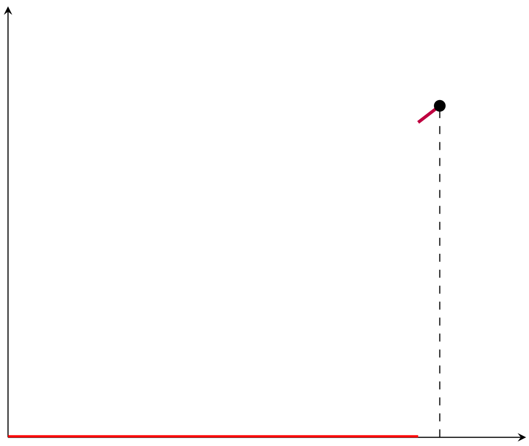
Set $y(x) \sim y_0(x) + \epsilon y_1(x) + \cdots$. Substitute into equation (2.1.1) and we have

$$\epsilon(y_0''(x) + \epsilon y_1''(x) + \cdots) + 2(y_0'(x) + \epsilon y_1'(x) + \cdots) + 2(y_0(x) + \epsilon y_1(x) + \cdots) = 0.$$

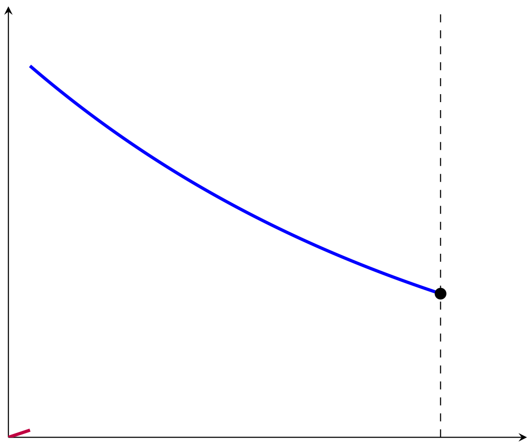
Balance $\mathcal{O}(1)$ and it provides

$$y_0' + y_0 = 0 \implies y_0(x) = ae^{-x}.$$

It leads to dilemma that the solution has only one arbitrary constant but we have two boundary conditions. It is over-determined. Moreover, the outer solution cannot describe solution over the whole domain $[0, 1]$. The following question is which boundary layer would we use?



(a) Possible boundary layer at $x = 1$



(b) Possible boundary layer at $x = 0$

Figure 2.1: Two choices of boundary layers. It can be chosen by investigating the sign of y'' near the boundary layer by looking at the concavity of the function.

2.1.2 Boundary layer

Assume that boundary layer is at $x = 0$. Introduce the stretched coordinate $\tilde{x} = x/\epsilon^\alpha$, $\alpha > 0$. Treat $\tilde{x}$ as fixed when ϵ is reduced. Setting $Y(\tilde{x}) = y(x)$ yields

$$\epsilon^{1-2\alpha}\frac{d^2Y}{d\tilde{x}^2} + 2\epsilon^{-\alpha}\frac{dY}{d\tilde{x}} + 2Y = 0, \quad Y(0) = 0. \tag{2.1.2}$$

Try a solution of a form

$$Y(\tilde{x}) \sim Y_0(\tilde{x}) + \epsilon^\gamma Y_1(\tilde{x}) + \cdots, \quad \gamma > 0.$$

Substitution into inner equation provides

$$\underbrace{\epsilon^{1-2\alpha}\frac{d^2}{d\tilde{x}^2}(Y_0 + \epsilon^\gamma Y_1 + \cdots)}_{(1)} + \underbrace{2\epsilon^{-\alpha}\frac{d}{d\tilde{x}}(Y_0 + \epsilon^\gamma Y_1 + \cdots)}_{(2)} + \underbrace{2(Y_0 + \epsilon^\gamma Y_1 + \cdots)}_{(3)} = 0.$$

One need to determined correct balance condition:

- Balance (1) and (3) with taking (2) is higher order. Then it requires $\alpha = 1/2$. Then (1), (3) $= \mathcal{O}(1)$, but (3) $= \mathcal{O}\epsilon^{-1/2}$. ($\times$)
- Balancing (2) and (3) gives outer solution. ($\times$)
- Balance (1) and (2) with taking (3) is higher order. Then it requires $\alpha = 1$. Then (1), (2) $= \mathcal{O}(\epsilon^{-1})$ and (3) $= \mathcal{O}(1)$. (Yay!)

Choosing the last balance, one can obtain an equation from $\mathcal{O}(\epsilon^{-1})$ terms

$$Y_0'' + 2Y_0' = 0, \quad 0 < \tilde{x} < \infty.$$

One can get inner solution $Y_0(\tilde{x}) = A(1 - e^{-2\tilde{x}})$, where A is unknown constant.

2.1.3 Matching

It remains to determine the constant A. The inner and outer solutions are both approximations of the same function. Hence they sould agree in the transition zone between inner and outer layers. Thus

$$\lim_{\tilde{x}\to\infty} Y_0(\tilde{x}) = \lim_{x\to 0^+} y_0(x), \tag{2.1.3}$$

and yields $A = e$. Therefore, $Y_0(x) = e(1 - e^{1-2x/\epsilon})$.

2.1.4 Composite expression

So far, we have a solution in two pieces, neither is uniformly valid in $x \in [0, 1]$. We would like to construct a composite solution that holds everywhere. One way is subtracting constant to match each one

$$Y(x) \sim y_0(x) + Y(x/\epsilon) - y_0(0). \tag{2.1.4}$$

Near $x = 0$, $y_0(x)$ is canceled out with the constant and vice versa.

The matching condition $Y_0(+\infty) = y_0(0^+)$ may not work in general. First, the limits might not exist. Second, complication may arise when constructing second order terms. A more general approach is to explicitly introduce an intermediate region between inner and outer domain. Introduce an intermediate variable $x_\eta = x/\eta(\epsilon)$ with $\epsilon \ll \eta \ll 1$. The inner and outer solution should give same result when expression in terms of x_η. Then

1. change from x to x_η in outer expansion $y_{\text{outer}}(x_\eta)$. Assume there is $\eta_1(\epsilon)$ such that y_{outer} is valid for $\eta_1(\epsilon) \ll \eta(\epsilon) \leq 1$.

2. Change variable $\tilde{x}$ to x_η in inner expansion to obtain $y_{\text{inner}}(x_\eta)$. Assume there is $\eta_2(\epsilon)$ such that inner is valid for $\epsilon \ll \eta(\epsilon) \ll \eta_2(\epsilon)$.

3. If $\eta_1 \ll \eta_2$, then domain of validity overlap (because inner expansion valid on $x \leq \eta_2$ and outer expansion valid on $x \geq \eta_2$) and we require $y_{\text{outer}} \sim y_{\text{innter}}$ in the overlap region.

Return to our particular example. Let $x_\eta = x/\epsilon^\beta$ with $0 < \beta < 1$. Then

$$y_{\text{inner}} \sim A(1 - e^{-2x_\eta/\epsilon^{1-\beta}}) \sim A + \mathcal{O}(\epsilon^{\beta-1}),$$

and

$$y_{\text{outer}} \sim e^{1-x_\eta\epsilon^\beta} \sim e + \mathcal{O}(\epsilon^\beta).$$

These are hard to match so we consider higher-order term. Find the second balance equation

$$y_1'' + y_1 = -\frac{1}{2}y_0'', \quad y_1(1) = 0 \quad\Longrightarrow\quad y_1(x) = \frac{1}{2}(1-x)e^{1-x}$$

from $\mathcal{O}(\epsilon)$ terms of outer expansion and

$$Y_1'' + 2Y_1' = -2Y_0, \quad Y_1(0) = 0 \quad\Longrightarrow\quad Y_1(\tilde{x}) = B(1 - e^{-2\tilde{x}}) - \tilde{x}e(1 + e^{-2\tilde{x}})$$

from $\mathcal{O}(1)$ terms of inner expansion. Determine B by matching on intermediate zone

$$y_{\text{outer}} \sim e^{1-x_\eta\epsilon^\beta} + \frac{\epsilon}{2}(1 - x_\eta\epsilon^\beta)e^{1-x_\eta\epsilon^\beta}$$

$$
\begin{aligned}
&\sim e\cdot 1 - e\cdot x_\eta \epsilon^\beta + \frac{\epsilon}{2}\cdot e\cdot 1 + e\cdot \frac{1}{2}x_\eta^2\epsilon^{2\beta} + \cdots \\
y_{\text{inner}} &\sim e(1-e^\xi) + \epsilon\left(B(1-e^\xi) - \frac{x_\eta}{\epsilon^{1-\beta}}e(1+e^\xi)\right), \quad \xi = -2x_\eta/\epsilon^{1-\beta} \\
&\sim e - \epsilon^\beta x_\eta \cdot e + \epsilon\cdot B + \cdots,
\end{aligned}
$$

and yields $B = e/2$. Therefore, the composite solution is

$$
y(x) \sim y_0(x) + \epsilon y_1(x) + Y_0(x/\epsilon) + \epsilon Y_1(x/\epsilon) - \left(e - \underbrace{x_\eta \epsilon^b}_{=x} e + \frac{e}{2}\cdot\epsilon\right). \tag{2.1.5}
$$

Remark 2.1.1. Things to look for in more general problems on $[0,1]$

1. The boundary layer could be at $x=1$ or there could be boundary layers at both ends. At $x=1$, the stretched coordinate is $\tilde{x} = (x-1)/\epsilon^\alpha$.
2. There is an interior layers at some $x_0(\epsilon)$

$$
\tilde{x} = \frac{x - x_0}{\epsilon^\alpha}.
$$

3. ϵ-dependence could be funky, e.g. $\nu = -1/\log\epsilon$.
4. The solution odes not have layered structure.

2.2 Extensions: multiple boundary layers, etc.

2.2.1 Multiple boundary layers

Consider a boundary value problem

$$
\epsilon^2 y'' + \epsilon x y' - y = -e^x, \quad \text{with } y(0) = 2, \text{ and } y(1) = 1, \tag{2.2.1}
$$

which is singular and non-linear. Note that in case when $\epsilon = 0$, we get $y = e^x$ and it does not match any boundary conditions. This solution is the first term in the outer solution, i.e. $y_0(x) = e^x$.

Start to find inner solution at $x=0$. Set $\tilde{x} = x/\epsilon^\alpha$ and $Y(\tilde{x}) = y(x)$. Then we have

$$
\underbrace{\epsilon^{2-2\alpha}\frac{d^2}{d\tilde{x}^2}Y}_{(1)} + \underbrace{\epsilon\tilde{x}\frac{d}{d\tilde{x}}Y}_{(2)} - \underbrace{Y}_{(3)} = -e^{-\tilde{x}\epsilon^\alpha} = \underbrace{-(1+\tilde{x}\epsilon^\alpha + \cdots)}_{(4)}.
$$

In order to balance (1),(3) and (4), we require $\alpha = 1$. Then taking $Y \sim Y_0 + \cdots$ yields the following balance equation for $\mathcal{O}(1)$

$$
Y_0'' - Y_0 = -1, \quad Y_0(0) = 2.
$$

Its general solution is

$$
Y_0(\tilde{x}) = 1 + Ae^{-\tilde{x}} + (1-A)e^{\tilde{x}}, \quad 0 < \tilde{x} < \infty.
$$

To achieve A, matching $Y_0(+\infty) = y_0(0)$ implies $A = 1$. At $x = 1$, setting $\tilde{x} = (x-1)/\epsilon^\beta$ and $\overline{Y}(\tilde{x}) = y(x)$ provides that

$$\epsilon^{2-2\beta}\frac{d^2}{d\tilde{x}^2}\overline{Y} + (1+\epsilon^\beta\tilde{x})\epsilon^{1-\beta}\frac{d}{d\tilde{x}}\overline{Y} - \overline{Y} = e^{-1+\epsilon^\beta\tilde{x}}.$$

Achieve balance for $\beta = 1$ and we obtain

$$\overline{Y}_0'' + \overline{Y}_0' - \overline{Y}_0 = -e, \quad -\infty < \tilde{x} < 0, \quad \text{with } \overline{Y}_0(0) = 1.$$

Its general solution is

$$\overline{Y}_0(\tilde{x}) = e + Be^{r_+\tilde{x}} + (1-e-B)e^{r_-\tilde{x}},$$

where $r_\pm = (-1 \pm \sqrt{5})/2$. Matching $\overline{Y}_0(-\infty) = y_0(1)$ provides $B = 1-e$. Therefore, its composite solution is

$$\begin{aligned} y &\sim y_0(x) + \left[Y_0\left(\frac{x}{\epsilon}\right) - Y_0(+\infty)\right] + \left[\overline{Y}_0\left(\frac{x}{\epsilon}\right) - \overline{Y}_0(-\infty)\right] \\ &\sim e^x + e^{-x/\epsilon} + (1-e)e^{-r_+(1-x)/\epsilon}. \end{aligned}$$

2.2.2 Interior layers

It is also possible for a boundary layer to occur in the interior of the domain rather than at a physical boundary – matching now has to determined the location at the interior layer. Consider a boundary value problem

$$\epsilon y'' = y(y'-1), \quad 0 < x < 1 \tag{2.2.2}$$

with $y(0) = 1$ and $y(1) = -1$. For its outer equation, setting $y \sim y_0 + \cdots$ yields

$$y_0(y_0'-1) = 0 \quad \Longrightarrow \quad y_0 = 0 \text{ or } y_0(x) = x + a$$

for some constant a. Since the outer equation does not satisfy both boundary condition at once, we need to find a boundary layer to fit boundary conditions.

Assume that boundary layer is at $x = 0$. In the boundary layer, $y'' > 0$ and $y' < 0$. Since y can be positive, we cannot match signs of differential equation everywhere. If boundary layer is at $x = 1$, then $y'' < 0$ and $y' - 1 < 0$. Since y can be negative, it cannot match signs everywhere in boundary layer. What if it has interior layer at $x = x_0$? For $x - x_0 = 0^-$, we have $y'' < 0$, $y' - 1 > 0$, but $y > 0$. For $x - x_0 = 0^+$, we have $y'' > 0$, $y' - 1 < 0$, but $y < 0$. Thus interior layer can match the signs.

From the argument of interior layer argument, find inner solution by setting $\tilde{x} = (x - x_0)/\epsilon^\alpha$, $0 < x_0 < 1$. Then we have two outer regions $0 \le x < x_0$ and $x_0 < x \le 1$. The inner equation is

$$\epsilon^{1-2\alpha}Y'' = \epsilon^{-\alpha}YY' - Y,$$

and one can balance if $\alpha = 1$. Setting $Y(\tilde{x}) \sim Y_0(\tilde{x}) + \cdots$ gives

$$Y_0'' = Y_0Y_0' \quad \Longrightarrow \quad Y_0' = \frac{1}{2}Y_0^2 + A.$$

It has three general solution depending on sign of A:

1. $Y_0 = B\left[\frac{1-De^{B\tilde{x}}}{1+De^{B\tilde{x}}}\right]$ if $A > 0$,
2. $Y_0 = B\tan\left[C - \frac{B\tilde{x}}{2}\right]$ if $A < 0$,
3. $Y_0 = \frac{2}{C-\tilde{x}}$ if $A = 0$.

Three forms (rather than a single general solution) reflects non-linearity.

Next, match the inner solution with outer solution

$$y_0(x) = \begin{cases} x+1 & , \quad x < x_0 \\ x-2 & , \quad x_0 < x \end{cases}.$$

Only inner solution 1. can match these outer solutions. Without loss of generality, assume $B > 0$ and we get

$$-B = Y_0(+\infty) = y_0(x_0^+) = x_0 - 2 \quad \text{and} \quad B = Y_0(-\infty) = y_0(x_0^-) = x_0 + 1.$$

It yields $x_0 = 1/2$ and $B = 3/2$. What about D? Remember that $y(x_0) = 0$. This implies that

$$Y_0(x_0) = 0 = \frac{3}{2}\cdot\frac{1-D}{1+D} \implies D = 1.$$

Therefore,

$$Y_0(\tilde{x}) \sim \frac{3}{2}\cdot\frac{1-e^{3\tilde{x}/2}}{1+e^{3\tilde{x}/2}}.$$

Finally, the composite solution can be constructed in the two domains $[0, x_0)$ and $(x_0, 1]$

$$y(x) \sim \begin{cases} x+1+\frac{3}{2}\cdot\frac{1-e^{3(2x-1)/4\epsilon}}{1+e^{3(2x-1)/4\epsilon}} - \frac{3}{2} & , \quad 0 \le x < x_0 \\ x-2+\frac{3}{2}\cdot\frac{1-e^{3(2x-1)/4\epsilon}}{1+e^{3(2x-1)/4\epsilon}} + \frac{3}{2} & , \quad x_0 < x \le 1 \end{cases}.$$

2.3 Partial differential equations

Consider Burger's equation

$$u_t + u\cdot u_x = \epsilon u_{xx} \quad , \quad -\infty < x < \infty,\ t > 0 \tag{2.3.1}$$

$$u(x,0) = \phi(x) \tag{2.3.2}$$

Notice that this perturbation problem is singular because type of solution is changed from parabolic to hyperbolic when $\epsilon > 0$ to $\epsilon = 0$. Assume that $\phi(x)$ is smooth and bounded except for a jump continuity at $x = 0$ with $\phi(0^-) > \phi(0^+)$ and $\phi' \ge 0$. For concreteness, set

$$u(x,0) = \begin{cases} 1, & x < 0 \\ 0, & 0 < x \end{cases}.$$

This is an example of a Riemann problem – evolves into a traveling front that sharpens as $\epsilon \to 0$. We can handle it in similar way for boundary layer problem. For outer solution, expanding $u(x,t) \sim u_0(x,t) + \cdots$ gives balance equation for $\mathcal{O}(1)$ terms

$$\partial_t u_0 + u_0\cdot\partial_x u_0 = 0.$$

Solve it using the *method of characteristics*

$$\frac{dt}{d\tau} = 1, \quad \frac{dx}{d\tau} = u_0 \quad \text{and} \quad \frac{du_0}{d\tau} = 0,$$

and it yields characteristic straight lines

$$x = x_0 + \phi(x_0)t.$$

Characteristic into set at the shock $x = s(t)$ with determined using the *Rankine-Hugoniot equation*

$$\dot{s} = \frac{1}{2} \cdot \frac{[\phi(x_0^+)]^2 - [\phi(x_0^-)]^2}{\phi(x_0^+) - \phi(x_0^-)} = \frac{1}{2}[\phi(x_0^+) + \phi(x_0^-)]. \tag{2.3.3}$$

We will derive an equation for $s(t)$ using match asymptotics. Introduce a moving inner layer around $s(t)$

$$\tilde{x} = \frac{x - s(t)}{\epsilon^\alpha}.$$

The inner PDE for $U(\tilde{x}, t) = u(x, t)$

$$\partial_t U - \epsilon^{-\alpha} s'(t)\partial_{\tilde{x}} U + \epsilon^{-\alpha} U \cdot \partial_{\tilde{x}} U = \epsilon^{1-2\alpha}\partial_{\tilde{x}}^2 U.$$

In order to balance terms, require $\alpha = 1$ and $U \sim U_0 + \cdots$

$$-s'(t)\partial_{\tilde{x}} U_0 + U_0 \cdot \partial_{\tilde{x}} U_0 = \partial_{\tilde{x}}^2 U_0.$$

Integrating with respect to $\tilde{x}$ gives

$$\partial_{\tilde{x}} U_0 = \frac{1}{2} U_0^2 - s'(t)U_0 + A(t).$$

Its matching conditions are

$$\lim_{\tilde{x}\to-\infty} U_0 = u_0^- \quad \text{and} \quad \lim_{\tilde{x}\to+\infty} U_0 = u_0^+$$

where $u_0^\pm = \lim_{x\to s(t)^\pm} u_0(x, t)$. Since $U_0(\tilde{x}, t)$ is a constant for $\tilde{x} \to \pm\infty$, we have $\partial_{\tilde{x}} U_0 \to 0$ as $\tilde{x} \to \pm\infty$. Then we have

$$0 = \frac{1}{2}[u_0^-]^2 - s'(t)u_0^- + A(t),$$
$$0 = \frac{1}{2}[u_0^+]^2 - s'(t)u_0^+ + A(t).$$

Subtracting part of equations yields

$$s'(t) = \frac{1}{2} \cdot \frac{[\phi(x_0^+)]^2 - [\phi(x_0^-)]^2}{\phi(x_0^+) - \phi(x_0^-)} = \frac{1}{2}[\phi(x_0^+) + \phi(x_0^-)].$$

Hence $A(t) = \frac{1}{2}u_0^+ u_0^-$. We now note that the inner equation can be rewritten as

$$\partial_{\tilde{x}} U_0 = \frac{1}{2}(U_0 - u_0^+)(U_0 - u_0^-),$$

with $u_0^\pm = u_0^\pm(t)$. Then one can achieve the following equations

$$\begin{aligned}\int dU_0 \left[\frac{1}{U_0 - u_0^+} - \frac{1}{U_0 - u_0^-}\right] &= \frac{1}{2}\int d\tilde{x}(u_0^+ - u_0^-)\\ \implies \log\left|\frac{U_0 - u_0^+}{U_0 - u_0^-}\right| &= \frac{1}{2}(u_0^+ - u_0^-)\tilde{x} + C(t)\\ \implies \frac{U_0 - u_0^+}{u_0^- - U_0} &= b(\tilde{x}, t) = B(t)e^{(u_0^+ - u_0^-)\tilde{x}/2}\end{aligned}$$

where $B(t) = e^{C(t)}$. Therefore,

$$U_0(\tilde{x}, t) = \frac{u_0^+ + b(\tilde{x}, t)u_0^-}{1 + b(\tilde{x}, t)}.$$

In order to determine $B(t)$, we have to go to next order [See Holmes for more details]. You may find, in the end,

$$B(t) = \sqrt{\frac{1 + t\phi'(x_0^+)}{1 + t\phi'(x_0^-)}}.$$

2.4 Strongly localized perturbation theory

This work is mainly done by Michael J. Ward, see [SWF07; BEW08; CSW09; Pil+10; Kur+15] for more details. Consider a diffusion equation with small holes. Before we start to apply perturbation theory on the problem, recall Green's function in two and three dimensional space. Green's function is solution with single input data, especially in case of Laplace operator,

$$\triangle u = \delta(x - x_0), \quad \text{in } \mathbb{R}^n, \quad n = 2, 3.$$

Then $u \sim -1/4\pi|x - x_0|$ as $x \to x_0$ in 3D and $u \sim \log|x - x_0|/2\pi$ as $x \to x_0$ in 2D. In case of 3D, the Laplace operator in spherical coordinate with angular symmetry is

$$\triangle u = u_r r + \frac{2}{r}u_r, \quad \text{for } |x - x_0| > 0,$$

and its solution is $u(r) = B/r$ for some constant r. By taking integral in Ω_ϵ, ball centered at x_0 with radius ϵ, we get

$$\int_{\Omega_\epsilon} \triangle u dx = \int_{\partial\Omega_\epsilon} \nabla u \cdot n ds = 4\pi r^2 \cdot u_r = -4\pi B = \int_{\Omega_\epsilon} \delta(x - x_0)dx = 1.$$

It yields that $u(r) = -1/4\pi r$, which is the Green's function in 3D.

2.4.1 Eigenvalue asymptotics in 3D

Let Ω be a 3D bounded domain with a hole of "radius" $\mathcal{O}(\epsilon)$, denoted by Ω_ϵ, removed from Ω. Consider an eigenvalue problem in $\Omega\backslash\Omega_\epsilon$ as follows:

$$\begin{cases} \triangle u + \lambda u = 0, & \text{in } \Omega\backslash\Omega_\epsilon \\ u = 0, & \text{on } \partial\Omega \\ u = 0, & \text{on } \partial\Omega_\epsilon \\ \int_{\Omega\backslash\Omega_\epsilon} u^2 dx = 1 \end{cases}. \tag{2.4.1}$$

We assume that Ω_ϵ shrinks to a point x_0 as $\epsilon \to 0$. For example, we could assume Ω_ϵ to be the sphere $|x - x_0| \le \epsilon$. The unperturbed problem is

$$\begin{cases} \triangle\phi + \lambda\phi = 0, & \text{in } \Omega \\ \phi = 0, & \text{on } \partial\Omega \ . \\ \int_{\Omega\setminus\Omega_\epsilon} \phi^2 dx = 1 \end{cases} \tag{2.4.2}$$

Assume this has eigenpair $\phi_j(x)$ and μ_j for $j = 0, 1, \cdots$ with $\int_\Omega \phi_j\phi_k dx = 0$ if $j \ne k$ and $\phi_0(x) > 0$ for $x \in \Omega$. We look for perturbed eigenpair near the $\phi_0(x)$ and μ_0. Expand $\lambda \sim \mu_0 + \nu(\epsilon)\lambda_1 + \cdots$ where ($\nu(\epsilon) \to 0$ as $\epsilon \to 0$.) In the outer region away from the hole, we take $u \sim \phi_0(x) + \nu(\epsilon)u_1(x) + \cdots$. Since $\Omega_\epsilon \to \{x_0\}$ as $\epsilon \to 0$, we have the following

$$\begin{cases} \triangle u_1 + \mu_0 u_1 = -\lambda_1\phi_0, & \text{in } \Omega\setminus\{x_0\} \\ u_1 = 0, & \text{on } \partial\Omega \qquad . \\ \int_\Omega 2u_1\phi_0 dx = 0 \end{cases} \tag{2.4.3}$$

Construct the inner solution near the hole. Let $y = (x - x_0)/\epsilon$ and set $V(x;\epsilon) = u(x_0 + \epsilon y)$. Then we find that V satisfies

$$\triangle_y V + \lambda\epsilon^2 V = 0, \qquad \text{outside of } \Omega_0 = \Omega_\epsilon/\epsilon.$$

Take $V \sim V_0 + \nu(\epsilon)V_1 + \cdots$ and get

$$\begin{cases} \triangle_y V_0 = 0, & \text{outside } \Omega_0 \\ V_0 = 0, & \text{on } \partial\Omega_0 \qquad . \\ V_0 \to \phi_0(x_0) \text{ as } |y| \to \infty \end{cases}$$

Try a solution of it by $V_0 = \phi_0(x_0)(1 - V_c(y))$. Then V_c satisfies

$$\begin{cases} \triangle_y V_c = 0, & \text{outside } \Omega_0 \\ V_c = 1, & \text{on } \partial\Omega_0 \qquad . \\ V_c \to 0 \text{ as } |y| \to \infty \end{cases}$$

A classical result from PDE theory is $V_c \sim C/|y|$ as $|y| \to \infty$ where C is electrostatic capacitance of Ω_0, determined by shape and size of Ω_0. We now have

$$V_0(x) \sim \phi_0(x_0)\left[1 - \frac{\epsilon C}{|x - x_0|}\right].$$

It has to match

$$\phi_0(x_0) + \nu(\epsilon)u_1$$

as $x \to x_0$. This yields that $\nu(\epsilon) = \epsilon$ and $u_(x) \to -\phi_0(x_0)C/|x - x_0|$ as $x \to x_0$. To evaluate perturbed eigenvalue λ_1, return to equation (2.4.3). Since $u_1 \to 4\pi\phi_0(x_0)C \cdot (-1/4\pi|x - x_0|)$ as $x \to x_0$, then we have the modified problem

$$\begin{cases} Lu_1 \equiv \triangle u_1 + \mu u_1 = -\lambda_1\phi_0 + 4\pi C\phi_0(x_0)\delta(x - x_0), & \text{in } \Omega \\ u_1 = 0, & \text{on } \partial\Omega \end{cases} . \tag{2.4.4}$$

Use Green's identity

$$\int_\Omega \phi_0 L u_1 - u_1 L\phi_0 dx = \int_{\partial\Omega} \phi_0 \partial_n u_1 - u_1 \partial_n \phi_0 ds.$$

Since $\phi_0 = u_1 = 0$ on $\partial\Omega$ and $L\phi_0 = 0$, we have

$$0 = \int_\Omega \phi_0 L u_1 dx = \int_\Omega \phi_0[-\lambda_1\phi_0 + 4\pi C\phi_0(x_0)\delta(x-x_0)]dx,$$

and it yields

$$\lambda_1 = \frac{4\pi C\phi_0^2(x_0)}{\int_\Omega \phi_0^2 dx}.$$

Therefore, $\lambda \sim \mu_0 + \epsilon\lambda_1$.

Remark 2.4.1. 1. Let us assume that $u = 0$ on $\partial\Omega$ is replaced by the no-flux condition on $\partial\Omega$. Then $\epsilon = 0$ problem becomes

$$\begin{cases} \triangle\phi + \mu\phi = 0, & \\ \partial_n\phi = 0, & \partial\Omega \ . \\ \int_\Omega \phi^2 dx = 1 & \end{cases}$$

The principal eigenvalues $\mu_0 = 0$ and $\phi_0(x) = 1/|\omega|^{1/2}$. In this case, $\lambda_1 \sim 4\pi C\epsilon/|\Omega|$ (to leading order it is independent of location x_0.)

2. For multiple holes Ω_{ϵ_j} for $j = 1, \cdots n$ and well-separated, its eigenvalue expansion is $\lambda \sim \mu_0 + 4\pi\epsilon \sum_j c_j[\phi_0(x_j)]^2 / \int_\Omega \phi_0^2 dx$.

2.4.2 Eigenvalue asymptotics in 2D

In the same fashion with 3D case, we want to find an asymptotic expansion of the same problem (2.4.1), but 2D. Let μ_0 and ϕ_0 be principal eigenpair of unperturbed problem (2.4.2). Set $\lambda \sim \mu_0 + \nu(\epsilon)\lambda_1 + \cdots$ for eigenvalue and $u \sim \phi_0 + \nu(\epsilon)u_1 + \cdots$ in outer region, where $\nu(\epsilon) \to 0$ as $\epsilon \to 0$. Then the equation for second term of outer expansion is (2.4.3). In the inner region, set $y = (x - x_0)/\epsilon$ and take $u(x) = \nu(\epsilon)V_0(y)$ where $\triangle_y V_0 = 0$. We want $V_0(y) \sim A_0 \log|y|$ as $|y| \to \infty$. To do so, setting $V_0 = A_0 V_c$ where

$$\begin{cases} \triangle_y V_c = 0, & y \in \Omega_0 \\ V_c = 0, & y \text{ on } \partial\Omega_0 \end{cases} \tag{2.4.5}$$

gives that

$$V_c \sim \log|y| - \log d + \mathcal{O}(1/|y|), \qquad \text{as } |y| \to \infty$$

where d is logarithmic capacitance determined by shape of Ω_0. It is interesting enough to notice the logarithmic capacitance of simple objects in the table. Then write inner solution in outer variable

$$u(x) \sim \nu(\epsilon)A_0 \log\frac{|y|}{d} \sim \nu(\epsilon)A_0\left[-\log(\epsilon d) + \log|x - x_0|\right].$$

Matching solution yields that

$$\phi_0(x_0) + \nu(\epsilon)u_1(x) \sim -\log(\epsilon d)A_0\nu(\epsilon) + A_0\nu(\epsilon)\log|x - x_0|,$$

Ω_0	**Geometric info**	**Capacitance** d
Circle	radius a	a
Ellipse	radius a, b	$(a+b)/2$
Triangle	side h	$\sqrt{3}[\Gamma(1/3)]^3 h/(8\pi^2)$
Rectangle	side h	$[\Gamma(1/4)]^2 h/(4\pi^{3/2})$

Table 2.1: The logarithmic capacitance in 2D for simple geometric figures.

as $x \to x_0$. In order to match the conditions, set $\nu(\epsilon) = -1/\log(\epsilon d)$. Then unknown constant $A_0 = \phi_0(x_0)$. Thus,

$$u_1(x) \sim \phi_0(x_0)\log|x - x_0|,$$

as $x \to x_0$. Hence, by the same procedure in 3D case by using Green's identity, one can find eigenvalue expansion

$$\lambda \sim \mu_0 + 2\pi \cdot \nu(\epsilon)\frac{[\phi_0(x_0)]^2}{\int_\Omega \phi_0^2 dx}.$$

Remark 2.4.2. Further terms in expansion yields

$$\lambda \sim \mu_0 + A_1\nu + A_2\nu^2 + A_3\nu^3 + \cdots.$$

Its potential problem is that the log decreases very slowly as ϵ decreases. Then the remaining term is quite large and break the asymptotic expansions. By summing the log series, one can solve the problem.

2.4.3 Summing all logarithmic terms

Consider Poisson's equation in a domain with one small hole given

$$\begin{cases} \triangle w = -B, & \text{in } \Omega\backslash\Omega_\epsilon \\ w = 0, & \text{on } \partial\Omega \\ w = 0, & \text{on } \partial\Omega_\epsilon \end{cases}. \tag{2.4.6}$$

In the outer region, set $w(x;\epsilon) = w_0(x;\nu(\epsilon)) + \sigma(\epsilon)w_1(x;\nu(\epsilon)) + \cdots$ where $\nu(\epsilon) = -1/\log(\epsilon d)$ and $\sigma \ll \nu^k$ for any $k > 0$. It gives the outer equation

$$\begin{cases} \triangle w_0 = -B, & \text{in } \Omega\backslash\{x_0\} \\ w = 0, & \text{on } \partial\Omega \\ w \text{ is singular}, & \text{as } x \to x_0 \end{cases}. \tag{2.4.7}$$

In the inner region, set $y = (x-x_0)/\epsilon$ and $V(y;\epsilon) = w(x_0+\epsilon y;\epsilon)$. Expand $V(y;\epsilon) = V_0(y;\nu(\epsilon)) + \mu_0(\epsilon)V_1(y;\nu(\epsilon)) + \cdots$ where $\mu_0 \ll \nu^k$ for all $k > 0$. Then V_0 satisfies

$$\begin{cases} \triangle_y V_0 = 0, & \text{outside } \Omega_0 \\ V_0 = 0, & \text{on } \partial\Omega_0 \end{cases}. \tag{2.4.8}$$

The leading order matching condition is

$$\lim_{x\to x_0} w_0(x;\nu) \sim \lim_{|y|\to\infty} V_0(y;\nu).$$

Introduce an unknown function $\gamma = \gamma(\nu)$ with $\gamma(0) = 1$ and let $V_0(y;\nu) = \nu\gamma V_c(y)$. Then it follows that

$$\begin{cases} \triangle_y V_c = 0, & \text{outside } \Omega_0 \\ V_c = 0, & \text{on } \partial\Omega_0 \\ V_c \sim \log|y|, & \text{as } |y| \to \infty \end{cases}.$$

Thus, $V_c(y) \sim \log|y| - \log d + \mathcal{O}(1/|y|)$ for $1 \ll |y|$. In original coordinate,

$$V_0(y;\nu) \sim \gamma + \nu\gamma \log|x - x_0|.$$

Matching condition gives $w_0 \sim \nu\gamma \log|x - x_0| + \gamma$ as $x \to x_0$. So outer problem is

$$\begin{cases} \triangle w_0 = -B, & \text{in } \Omega\backslash\{x_0\} \\ w = 0, & \text{on } \partial\Omega \\ w \sim \gamma + \nu\gamma\log|x - x_0| & \text{as } x \to x_0 \end{cases}. \tag{2.4.9}$$

Introduce $w_{OH}(x)$ and $G(x;x_0$ with

$$\begin{cases} \triangle w_{OH} = -B, & \text{in } \Omega \\ w_{OH} = 0, & \text{on } \partial\Omega \end{cases}, \quad \begin{cases} \triangle G = \delta(x - x_0), & \text{in } \Omega \\ G = 0, & \text{on } \partial\Omega \end{cases}.$$

One can find $G(x;x_0) = \frac{1}{2\pi}\log|x-x_0|+R_0(x;x_0)$ where $R_0$is the regular prt of Green's function which converges as $x \to x_0$. Then we can write down the solution

$$w_0(x;\nu) = w_{OH}(x) + 2\pi\gamma\nu G(x;x_0).$$

As $x \to x_0$, we obtain the asymptotic condition

$$w_{OH}(x_0) + 2\pi\gamma\nu\left[\frac{1}{2\pi}\log|x - x_0| + R(x;x_0)\right] = \gamma + \gamma\nu\log|x - x_0|,$$

and it yields

$$\gamma(\nu) = \frac{w_{OH}(x_0)}{1 - 2\pi\nu R_0(x_0;x_0)}.$$

Therefore, the final expansion of the Poisson equation is

$$w(x) \sim w_{OH}(x) + \frac{\nu(\epsilon)}{1 - 2\pi R_0(x_0;x_0)\nu(\epsilon)} \cdot 2\pi w_{OH}(x_0)G(x;x_0).$$

2.5 Exercises

1. Find a composite expansion of the solution to the following problems on $x \in [0,1]$ with a boundary layer at the end $x = 0$:

   (a) $\epsilon y'' + 2y' + y^3 = 0$, $y(0) = 0$, $y(1) = 1/2$.

Solution: To find outer expansion y, set $y \sim y_0 + \cdots$ and balance $\mathcal{O}(1)$ terms

$$0 + 2y_0' + y_0^3 = 0.$$

It yields a general solution

$$y_0^{-2} = x + D$$

for some constant D. Since the boundary layer is at $x = 0$, boundary condition at $x = 1$ determines integrating constant D and yields

$$y_0(x) = \frac{1}{\sqrt{x+3}}.$$

Now, construct inner expansion near $x = 0$. Setting $\tilde{x} = x/\epsilon^\alpha$ and $Y(\tilde{x}) = y(x)$ yields ODE for inner solution

$$\epsilon^{1-2\alpha} Y'' + 2\epsilon^{-\alpha} Y' + Y^3 = 0.$$

In order to balance terms, require $\alpha = 1$ and $Y \sim Y_0 + \cdots$. Then we achieve

$$Y_0'' + 2Y_0' = 0.$$

A general solution of Y_0 is

$$Y_0(\tilde{x}) = C(1 - e^{-2\tilde{x}})$$

with boundary condition $Y_0(0) = 0$. Matching condition

$$\lim_{\tilde{x}\to\infty} Y_0(\tilde{x}) = \lim_{x\to 0^+} y_0(x)$$

yields $C = 1/\sqrt{3}$. Therefore, the composite expansion of the solution is

$$y(x) \sim y_0(x) + Y\left(\frac{x}{\epsilon}\right) - \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{x+3}} - \frac{1}{\sqrt{3}} e^{-2x/\epsilon}.$$

(b) $\epsilon y'' + (1 + 2x)y' - 2y = 0$, $y(0) = \epsilon$, $y(1) = \sin(\epsilon)$.

Solution: To find outer expansion y, set $y \sim y_0 + \epsilon y_1 + \cdots$ and balance $\mathcal{O}(1)$ terms

$$0 + (1 + 2x)y_0' - 2y' = 0, \quad y_0(0) = y_0(1) = 0,$$

and its general solution is $y_0(x) = C(2x + 1)$. Since we know that it has a boundary layer at $x = 0$, match boundary condition and get $C = 0$. Thus $y_0(x) = 0$. Balancing $\mathcal{O}(\epsilon)$ terms yields that

$$y_0'' + (1 + 2x)y_1' - 2y_1', \quad y_1(0) = y_1(1) = 1,$$

and its solution with boundary condition at $x = 1$ is $y_1(x) = (2x + 1)/3$. It follows that the outer expansion is

$$y(x) \sim \frac{\epsilon}{3}(2x+1) + \cdots .$$

Now, consider inner expansion near $x = 0$. Setting $\tilde{x} = x/\epsilon^{\alpha}$ and $Y(\tilde{x}) = y(x)$ yields ODE for inner solution

$$\epsilon^{1-2\alpha} Y' + \epsilon^{-\alpha} Y' + 2\tilde{x} Y' - 2Y = 0.$$

To balance the equation, it requires that $\alpha = 1$ by setting $Y \sim Y_0$. Then we have

$$Y_0'' + Y_0' = 0, \quad Y_0(0) = 0 \quad \Longrightarrow \quad Y_0(\tilde{x}) = D(1 - e^{-\tilde{x}}).$$

Matching condition gives

$$\lim_{\tilde{x}\to\infty} Y(\tilde{x}) = D = \lim_{x\to 0} y(x) = \frac{\epsilon}{3}.$$

Therefore, the composite expansion of the solution is

$$y(x) \sim \epsilon y_1(x) + (Y(x/\epsilon) - \epsilon/3) = \frac{\epsilon}{3}(2x + 1 - e^{-x/\epsilon}).$$

2. Consider the integral equation

$$\epsilon y(x) = -q(x) \int_0^x [y(s) - f(s)] s ds, \quad 0 \le x \le 1,$$

where $f(x)$ is positive and smooth.

(a) Taking $q(x) = 1$ find a composite expansion of the solution $y(x)$. [Hint: convert to an ODE.]

Solution: Observe that

$$\epsilon y(0) = 0 \quad \Longrightarrow \quad y(0) = 0.$$

Taking derivative on given integral equation gives

$$\epsilon y'(x) + x y(x) = x f(x).$$

One can get the outer expansion by setting $y(x) \sim y_0(x) + \cdots$

$$x y_0(x) = x f(x) \quad \Longrightarrow \quad y_0(x) = f(x), \quad 0 < x \le 1.$$

Since f is positive function $\lim_{x\to 0} y_0(x) = f(0) > 0$, which does not match boundary condition. It implies that the expansion has boundary layer at $x = 0$. Scale near $x = 0$ by taking a new coordinate $\tilde{x} = x/\epsilon^{\alpha}$ and $Y(\tilde{x}) = y(x)$. In this coordinate, the smooth function $f(x)$ can be count as a constant $f(x) \sim f(0)$. It follows the ODE for inner expansion

$$\epsilon^{1-\alpha} Y' + \epsilon^{\alpha} \tilde{x} Y = \epsilon^{\alpha} \tilde{x} f(0).$$

To balance the equation, it requires $1-\alpha=\alpha$, i.e. $\alpha=1/2$ and its general solution with boundary condition $Y(0)=0$ yields the first term inner expansion

$$Y(\tilde{x}) = f(0)\left(1-e^{-\tilde{x}^2/2}\right),$$

and it matches with outer solution

$$\lim_{\tilde{x}\to\infty} Y(\tilde{x}) = f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} y_0(x).$$

Therefore, the composite expansion of the integral equation is

$$y(x) \sim f(x) - f(0)e^{-x^2/2\epsilon}.$$

(b) Generalize to the case that $q(x)$ is a positive smooth function.

Solution: It is also true that $y(0)=0$ because f is positive function. Taking derivative and substituting integral term gives

$$\epsilon y' = \epsilon \frac{q'}{q} y - q(y-f)x,$$

and one can rewrite it as

$$\epsilon\left(\frac{y}{q}\right)' + xq\left(\frac{y}{q}\right) = xf \qquad \Longrightarrow \qquad \epsilon z' + xqz' = xf$$

by setting $z=y/q$. In the same fashion in part 1., obtain outer expansion by balancing $\mathcal{O}(1)$

$$z(x) \sim z_0(x) = \frac{f(x)}{q(x)} \qquad \Longrightarrow \qquad y(x) \sim y_0(x) = f(x).$$

Since f, q are positive, then z has boundary layer at $x=0$. With the same argument in part 1., one can get the ODE for inner expansion $Z(\tilde{x})$

$$Z' + xq(0)Z = xf(0) \qquad \Longrightarrow \qquad Z(\tilde{x}) = \frac{f(0)}{q(0)}\left(1-e^{-q(0)\tilde{x}^2/2}\right),$$

that is $Y(\tilde{x}) \sim f(0)\left(1-e^{-q(0)\tilde{x}^2/2}\right)$. Therefore, its composite expansion is

$$y(x) \sim f(x) - f(0)e^{-q(x)x^2/2\epsilon}.$$

(c) Show that solution of part 2. still holds if $q(x)$ is continuous but not differentiable everywhere on $[0,1]$.

Solution: The basic idea showing the claim is to derive all the expansions from integral equation. For the outer expansion, setting $y \sim y_0$ gives

$$0 = -q(x)\int_0^x (y_0(s) - f(s))sds \implies \int_0^x (y_0(s) - f(s))sds = 0$$

because $q(x)$ is positive. Without worrying about differentiability of q, take derivative on the equation and get same outer expansion $y_0(x) = f(x)$. In the similar way, to find the inner expansion, set the new coordinate $\tilde{x} = x/\epsilon^\alpha$ and $Y(\tilde{x}) = y(x)$. By approximating continuous function $q(x) = q(0)$ and $f(x) = f(0)$ in the boundary layer, it follows that

$$\epsilon^{1-\alpha}Y = -q(0)\int_0^{\epsilon^\alpha \tilde{x}} (Y(s/\epsilon^\alpha) - f(0))sds.$$

Now, one can take derivative and get the same differential equation for inner expansion. Therefore, one can achieve the same composite expansion.

3. (Boundary layer at both ends) Find a composite expansion of the following problem on $[0,1]$ and sketch the solution:

$$\epsilon y'' + \epsilon(x+1)^2 y' - y = x - 1, \quad y(0) = 0, \quad y(1) = -1.$$

Solution: To find outer expansion y, set $y \sim y_0 + \cdots$ and balance $\mathcal{O}(1)$ terms

$$0 = y_0 + x - 1 \implies y_0(x) = 1 - x.$$

It does not satisfy neither boundary conditions. Hence there are two boundary layer at $x = 0$ and $x = 1$. First, consider boundary layer at $x = 0$ by setting $\tilde{x} = x/\epsilon^\alpha$ for $\alpha > 0$ and $U(\tilde{x}) = y(x)$. It follows ODE for U

$$\epsilon^{1-2\alpha}U'' + (\epsilon^{1+\alpha}\tilde{x}^2 + \epsilon \cdot 2\tilde{x} + \epsilon^{1-\alpha})U' = U - 1 + \epsilon^\alpha \tilde{x}.$$

Since $\alpha > 0$, the smallest order of LHS is $\mathcal{O}(1-2\alpha)$ and RHS is $\mathcal{O}(1)$. To balance them, require $\alpha = 1/2$ and setting $U \sim U_0$ provides

$$U'' = U - 1 \implies U(\tilde{x}) = Ae^{\tilde{x}} + Be^{-\tilde{x}} + 1.$$

By boundary condition at $x = 0$, $y(0) = 0$, we achieve $A + B + 1 = 0$. Matching condition yields

$$\lim_{\tilde{x}\to\infty} U(\tilde{x}) = \lim_{x\to 0^+} y(x) = 1 \implies A = 0,$$

and $U(\tilde{x}) = 1 - e^{-\tilde{x}}$. Similarly, to find inner expansion at $x = 1$, set $\xi = (x-1)/\epsilon^\beta$ and $V(\xi) = y(x)$. It provides ODE for V

$$\epsilon^{1-2\beta}V'' + (\epsilon^{1+\beta}\tilde{x}^2 + \epsilon \cdot 4\tilde{x} + \epsilon^{1-\beta} \cdot 4)V' = V\epsilon^\beta \tilde{x}.$$

Since $\beta > 0$, the smallest order of LHS is $\mathcal{O}(1-2\alpha)$ and RHS is $\mathcal{O}(1)$. To balance them, require $\beta = 1/2$ and setting $U \sim U_0$ provides

$$V'' = V \qquad \Longrightarrow \qquad V(\xi) = Ce^{\xi} + De^{-\xi}.$$

By boundary condition at $x = 1$, that is $y(1) = -1$, we obtain $C+D = -1$. Matching outer and inner layer near $x = 1$ gives that

$$\lim_{\xi \to -\infty} V(\xi) = \lim_{x \to 1^-} y(x) = 0 \qquad \Longrightarrow \qquad D = 0,$$

and $V(\xi) = -e^{\xi}$. Therefore, the composite expansion of the solution is

$$y(x) \sim y_0(x) + \left[U\left(\frac{x}{\epsilon^{1/2}}\right) - 1\right] + \left[U\left(\frac{x-1}{\epsilon^{1/2}}\right) - 0\right],$$

and it follows that

$$y(x) \sim 1 - x - e^{-x/\epsilon^{1/2}} - e^{x-1/\epsilon^{1/2}}.$$

4. (Matched asymptotics can also be used in the time domain) The Michaelis-Menten reaction scheme for an enzyme catalyzed reaction is

$$\begin{aligned} \frac{ds}{dt} &= -s + (\mu + s)c, \\ \epsilon \frac{dc}{dt} &= s - (\kappa + s)c, \end{aligned}$$

where $s(0) = 1$, $c(0) = 0$. Here $s(t)$ is the concentration of substrate, $c(t)$ is the concentration of the catalyzed chemical product, and μ, κ are positive constants with $\mu < \kappa$. Find the first term in the expansions in the outer layer, the initial layer around $t = 0$, and the composite expansion.

Solution: Find the expansions in the outer layer by setting $s \sim s_0 + \cdots$ and $c \sim c_0 + \cdots$ and balancing $\mathcal{O}(1)$ terms

$$\begin{aligned} \frac{ds_0}{dt} &= -s_0 + (\mu + s_0)c_0, \\ 0 &= s_0 - (\kappa + s_0)c_0. \end{aligned}$$

It yields that

$$s_0(t) - 1 + \kappa \log s_0(t) = (\mu - \kappa)t, \quad c_0(t) = \frac{s_0(t)}{s_0(t) + \kappa}.$$

Notice that s_0 is implicitly determined. One can observe that c has a layer near $t = 0$. Setting $\tilde{t} = t/\epsilon^{\alpha}$, $S(\tilde{t}) = s(t)$ and $C(\tilde{t}) = c(t)$ gives the system of ODE

$$\begin{aligned} \epsilon^{-\alpha} \frac{dS}{d\tilde{t}} &= -S + (\mu + S)C, \\ \epsilon^{1-\alpha} \frac{dC}{d\tilde{t}} &= S - (\kappa + S)C \end{aligned}$$

It requires that $\alpha = 1$ to balance equation for C not same with outer expansion. By setting $S \sim S_0$ and $C \sim C_0 + \cdots$, it follows that

$$\begin{aligned} \frac{dS_0}{d\tilde{t}} &= 0, \\ \frac{dC_0}{d\tilde{t}} &= S_0 - (\kappa + S_0)C_0. \end{aligned}$$

First equation with initial condition $s(0) = 1$ gives that $S_0(\tilde{t}) = 1$. Hence we write ODE for C_0 as

$$\frac{dC_0}{d\tilde{t}} = 1 - (\kappa+1)C_0 \qquad \Longrightarrow \qquad C_0(\tilde{t}) = \frac{1}{\kappa+1}\left(1 - e^{-(\kappa+1)\tilde{t}}\right).$$

Fortunately, this solution satisfies matching condition

$$\lim_{\tilde{t}\to\infty} C_0(\tilde{t}) = \frac{1}{\kappa+1} = \frac{s_0(0)}{s_0(0)+\kappa} = \lim_{t\to 0} c_0(t).$$

Therefore, the composite solution of perturbation equation is

$$c(t) \sim c_0(t) - \frac{1}{\kappa+1}e^{-(\kappa+1)t/\epsilon}$$

where c_0 is implicitly determined by s_0.

5. (Implicit inner solution) A classical model in gas lubrication theory is the Reynolds equation

$$\epsilon\frac{d}{dx}\left(H^3 yy'\right) = \frac{d}{dx}(Hy), \quad 0 < x < 1,$$

where $y(0) = y(1) = 1$. Here $H(x)$ is a known, smooth, positive function with $H(0) \neq H(1)$.

(a) Suppose that there is a boundary layer at $x = 1$. Construct the first terms of the outer and inner solutions. Note that the leading order term Y_0 of the inner solution is defined implicitly according to $(x-1)/\epsilon = F(Y_0)$. Calculate the function F.

Solution: Setting $y \sim y_0$ and balancing $\mathcal{O}(1)$ terms yields outer solution equation

$$0 = \frac{d}{dx}(Hy_0) \implies y_0(x) = \frac{C}{H(x)}$$

where C is constant. Since we have boundary layer at $x = 1$, then applying boundary condition at $x = 0$ to outer solution gives $y_0(x) = H(0)/H(x)$. In the inner layer, setting $\tilde{x} = (x-1)/\epsilon^\alpha$ and $Y(\tilde{x}) = y(x)$ provides the ODE for inner solution

$$\epsilon^{1-2\alpha}\frac{d}{d\tilde{x}}\left(H^3 YY'\right) = \epsilon^{-\alpha}\frac{d}{d\tilde{x}}(HY).$$

Since the inner layer near $x = 1$, then continuous function H can be approximated as $H(x) \sim H(1)$. It follows that

$$\epsilon^{1-2\alpha} H^3(1) \frac{d}{d\tilde{x}}(Y_0 Y_0') = \epsilon^{-\alpha} H(1) \frac{d}{d\tilde{x}} Y_0.$$

for first expansion term Y_0 of Y. To balance the equation, it requires $\alpha = 1$ and now get

$$\frac{d}{d\tilde{x}}(Y_0 Y_0') = \frac{1}{H^2(1)} \frac{d}{d\tilde{x}} Y_0.$$

The general solution of ODE is given by

$$Y_0(\tilde{x}) - 1 - C \log \left| 1 + \frac{Y}{C} \right| = \frac{\tilde{x}}{H^2(1)}$$

with boundary condition $Y_0(0) = 1$. Matching condition gives

$$\lim_{\tilde{x}\to\infty} Y_0(\tilde{x}) = \lim_{x\to 1} y_0(x) = \frac{H(0)}{H(1)}.$$

It determines $C = -H(0)/H(1)$. Thus we have

$$F(Y_0) = H^2(1)(Y_0 - 1) + H(0)H(1) \log \left| 1 - \frac{H(1)Y_0}{H(0)} \right| = \tilde{x}.$$

(b) Use matching to construct the composite solution.

Solution: By the result from part 1., one can write the composite solution as

$$y(x) \sim \frac{H(0)}{H(x)} + \left[F^{-1}\left(\frac{x-1}{\epsilon}\right) - \frac{H(0)}{H(1)} \right].$$

(c) Show that if the boundary layer was assumed to be at $x = 0$, then the inner and outer solutions would not match.

Solution: It follows the same procedure in part 2., but achieve different F

$$F(Y_0) = H^2(0)(Y_0 - 1) + H(0)H(1) \log \left| 1 - \frac{H(0)Y_0}{H(1)} \right| = \tilde{x}.$$

However, as $\tilde{x} \to \infty$, the RHS tends to negative infinity. It does not match the conditions.

6. (Boundary layer at both ends) In a one-dimensional bounded domain, the potential $\phi(x)$ of an ionized gas satisfies

$$-\frac{d^2\phi}{dx^2} + h(\phi/\epsilon) = \alpha, \quad 0 < x < 1,$$

with boundary conditions

$$\phi'(0) = -\gamma, \quad \phi'(1) = \gamma.$$

Charge conservation requires

$$\int_0^1 h(\phi(x)/\epsilon)dx = \beta.$$

The function $h(s)$ is smooth and strictly increasing with $h(0) = 0$. The positive constants α and β are known (and independent of ϵ), and the constant γ is determined from the conservation equation.

(a) Calculate γ in terms of α and β.

Solution: Integration on given differential equation over $[0, 1]$ gives

$$-[\phi'(1) - \phi'(0)] + \int_0^1 h(\phi(x)/\epsilon)dx = \alpha \cdot 1 \quad \Longrightarrow -2\gamma + \beta = \alpha,$$

and it yields $\gamma = (\beta - \alpha)/2$.

(b) Find the exact solution for the potential when $h(s) = s$. Sketch the solution for $\gamma < 0$ and small ϵ, and describe the boundary layers that are present.

Solution: With $h(s) = s$, we have

$$-\frac{d^2\phi}{dx^2} + \frac{\phi}{\epsilon} = \alpha,$$

and its general solution is

$$\phi(x) = A\sinh\left(\frac{x}{\sqrt{\epsilon}}\right) + B\cosh\left(\frac{x}{\sqrt{\epsilon}}\right) + \epsilon\alpha,$$

where A, B are constants. Then one can obtain it derivative

$$\phi'(x) = \frac{1}{\sqrt{\epsilon}}\left[A\cosh\left(\frac{x}{\sqrt{\epsilon}}\right) + B\sinh\left(\frac{x}{\sqrt{\epsilon}}\right)\right]$$

To determine A and B, imposing boundary conditions to the general solution we have

$$\phi'(0) = \frac{A}{\sqrt{\epsilon}} = -\gamma,$$

and

$$\phi'(1) = \frac{1}{\sqrt{\epsilon}}\left[A\cosh\left(\frac{1}{\sqrt{\epsilon}}\right) + B\sinh\left(\frac{1}{\sqrt{\epsilon}}\right)\right] = \gamma$$

Solving them for A, B yields

$$A = -\gamma\sqrt{\epsilon}, \quad B = \gamma\sqrt{\epsilon} \cdot \frac{1 + \cosh(1/\sqrt{\epsilon})}{\sinh(1/\sqrt{\epsilon})}.$$

Then it follows that

$$\phi'(x) = \frac{\gamma}{\sinh(1/\sqrt{\epsilon})}\left[\sinh\left(\frac{x-1}{\sqrt{\epsilon}}\right) + \sinh\left(\frac{x}{\sqrt{\epsilon}}\right)\right].$$

For $x \neq 0, 1$, then $\phi'(x)$ decays to zero as $\epsilon \to 0$. Since $\phi'(0)$ and $\phi'(0)$ are nonzero, then it implies that ϕ has boundary layers at $x = 0, 1$.

(c) Suppose that $h(s) = s^{2k+1}$, where k is a positive integer, and assume $\beta < \alpha$. Find the first term in the inner and outer expansions of the solution.

Solution: With $h(s) = s^{2k+1}$, we have

$$-\epsilon^{2k+1}\frac{d^2\phi}{dx^2} + \phi^{2k+1} = \epsilon^{2k+1}\alpha, \tag{2.5.1}$$

with same boundary conditions. For $\epsilon = 0$, ϕ has a trivial solution. Thus we expand ϕ as

$$\phi \sim \epsilon^p(\phi_0 + \epsilon^q\phi_1 + \cdots),$$

and its derivatives are

$$\phi' \sim \epsilon^p(\phi_0' + \epsilon^q\phi_1' + \cdots), \quad \phi'' \sim \epsilon^p(\phi_0'' + \epsilon^q\phi_1'' + \cdots).$$

First, consider the boundary layer at $x = 0$. Rescale as $\tilde{x} = x/\epsilon^r$ and set $\Phi(\tilde{x}) = \phi(x)$. Then we have

$$\frac{d}{dx} \to \frac{d}{dx}\frac{d\tilde{x}}{dx} = \epsilon^{-r}\frac{d}{d\tilde{x}}.$$

It allows the governing equation in the boundary layer at $x = 0$ to be

$$-\epsilon^{2k+1-2r}\Phi'' + \Phi^{2k+1} = \epsilon^{2k+1}\alpha, \tag{2.5.2}$$

with the boundary condition

$$\epsilon^{-r}\Phi'(0) = -\gamma, \tag{2.5.3}$$

and it requires $r = p$ and gives $\Phi_0(0) = -\gamma$. Then (2.5.2) turns out to be

$$\begin{aligned}-\epsilon^{2k+1-p}(\Phi_0'' + \epsilon^q\Phi_1'' + \cdots) +& \\ \epsilon^{(2k+1)p}(\Phi_0 + \epsilon\Phi_1 + \cdots)^{2k+1} &= \epsilon^{2k+1}\alpha.\end{aligned}$$

To construct a boundary layer at $x = 0$, the only remaining case is to balancing $\mathcal{O}(\epsilon^{2k+1-p})$ and $\mathcal{O}(\epsilon^{(2k+1)p})$ and it requires $p = (2k+1)/(2k+2)$. Then we have a differential equation for boundary layer at $x = 0$

$$-\Phi_0'' + \Phi_0^{2k+1} = 0. \tag{2.5.4}$$

Multiplying Φ_0' and perform integration gives

$$-\frac{1}{2}(\Phi_0')^2 + \frac{\Phi_0^{2k+2}}{2k+2} = C,$$

for some constant C. As $\tilde{x} \to \infty$, Φ_0 matches with the outer solution $\phi(x) = 0$ for $0 < x < 1$. It implies that $C = 0$. Then we have its general solutions

$$\Phi_0(\tilde{x}) = \left[\frac{k}{\sqrt{k+1}}(\pm\tilde{x} - D)\right]^{-1/k},$$

for some constant D. Its derivative becomes

$$\Phi_0'(\tilde{x}) = -\frac{1}{k}\left[\frac{k}{\sqrt{k+1}}(\pm\tilde{x} - D)\right]^{-(k+1)/k} \cdot \left(\pm\frac{k}{\sqrt{k+1}}\right). \tag{2.5.5}$$

Imposing boundary condition at $x = 0$ gives

$$-\frac{1}{k}\left[-\frac{kD}{\sqrt{k+1}}\right]^{-(k+1)/k} \cdot \left(\pm\frac{k}{\sqrt{k+1}}\right) = -\gamma.$$

Since $\gamma = (\beta - \alpha)/2 < 0$, then we choose the negative sign and determine D such that

$$-\frac{kD}{\sqrt{k+1}} = (-\gamma\sqrt{k+1})^{-k/(k+1)} := \lambda.$$

Setting $\kappa = k/\sqrt{k+1}$ gives

$$\Phi_0(\tilde{x}) = (\lambda - \kappa\tilde{x})^{-1/k}. \tag{2.5.6}$$

Since the boundary layer at $x = 1$ satisfies the same differential equation (2.5.4), then one can derive the lowest order boundary layer Ψ_0 with rescaling $\hat{x} = (1-x)/\epsilon^r$

$$\Psi_0(\hat{x}) = (\lambda + \kappa\hat{x})^{-1/k}. \tag{2.5.7}$$

Therefore, the match asymptotic expansion of the differential equation is

$$y(x) \sim \left[\lambda - \frac{\kappa x}{\epsilon^r}\right]^{-1/k} + \left[\lambda + \frac{\kappa(1-x)}{\epsilon^r}\right]^{-1/k}, \tag{2.5.8}$$

where $r = (2k+1)/(2k+2)$.

(d) Can one construct a composite solution using the first terms?

Solution: Not exactly :)

7. (Internal boundary layer) Consider the problem

$$\epsilon y'' + y(1-y)y' - xy = 0, \quad 0 < x < 1,$$

with $y(0) = 2$ and $y(1) = -2$. A numerical solution for small ϵ shows that there is a boundary later at $x = 1$ and an internal layer at some x_0, where $y \sim 0$.

(a) Find the first term in the expansion of the outer solution. Assume that this function satisfies the boundary condition at $x = 0$.

Solution: Setting $y \sim y_0 + \cdots$ and balancing $\mathcal{O}(1)$ yields the following equation

$$y_0(1 - y_0)y_0' - xy_0 = 0.$$

Since we assume that it satisfies $y(0) = 2$, then $y_0(x) \neq 0$. Then it follows that

$$(1 - y_0)y_0' = x \qquad \Longrightarrow \qquad y_0(x) = 1 + \sqrt{1 - x^2}.$$

(b) Explain why there cannot be a boundary layer at $x = 1$, which links the boundary condition at $x = 1$ with the outer solution of part 1. evaluated at $x = 1$.

Solution: If it has a boundary layer at near $x = 1$, then the solution connects $\lim_{t \to 1} y_0(x) = 1$ and boundary condition $y(1) = -2$. Then there is x in the boundary layer such that $0 < y < 1$. Since $y'', y' < 0$ in the layer, then one can conclude

$$\epsilon y'' + y(1 - y)y' - xy < 0,$$

that is such expansion cannot satisfy IVP.

(c) Assume that there is an interior layer at some point x_0, which links the outer solution calculated in (a) for $0 \leq x < x_0$ with the outer solution $y \sim 0$ for $x_0 < x < 1$. From the matching show that $x_0 = \sqrt{3}/2$. Note that there will be an undetermined constant.

Solution: In the interior layer, scale the coordinate as $\tilde{x} = (x - x_0)/\alpha$ and set $Y(\tilde{x}) = y(x)$. Then one can achieve equation for the interior solution

$$\epsilon^{1-2\alpha} Y'' + \epsilon^{-\alpha} Y(1 - Y)Y' - (\epsilon^{\alpha}\tilde{x} + x_0)Y = 0.$$

Expanding the interior solution as $Y \sim Y_0 + \cdots$ and setting $\alpha = 1$ yields the balance equation for $\mathcal{O}(\epsilon^{-1})$ terms

$$Y'' + Y(1 - Y)Y' = 0.$$

Taking integration on both sides,

$$Y' + \frac{1}{2}Y^2 - \frac{1}{3}Y^3 = C,$$

where C is constant. From the right matching condition, $\lim_{\tilde{x} \to \infty} Y(\tilde{x}) = \lim_{\tilde{x} \to \infty} Y'(\tilde{x}) = 0$. Hence, $C = 0$. Invoking partial fraction to the separable ODE gives the general solution

$$\frac{1}{6}\tilde{x} + D = -\frac{2}{9}\log Y - \frac{1}{3Y} + \frac{2}{9}\log(3 - 2Y).$$

The left matching condition yields that

$$\lim_{\tilde{x}\to-\infty} Y(\tilde{x}) = \frac{3}{2} = \lim_{x\to x_0^-} y(x) = 1 + \sqrt{1 - x_0^2},$$

and it implies that $x_0 = \sqrt{3}/2$. With the undetermined constant D, the interior expansion $Y(\tilde{x}) = G^{-1}(\tilde{x})$ where

$$G(Y) = 6\left[-\frac{2}{9}\log Y - \frac{1}{3Y} + \frac{2}{9}\log(3-2Y) - D\right].$$

(d) Given the interior layer at x_0, construct the first term in the expansion of the inner solution at $x = 1$.

Solution: In the similar fashion, setting $\xi = (x-1)/\epsilon^\beta$ and $V(\xi) = y(x)$. Then we get the same ODE with the left matching condition

$$V'' + V(1-V)V' = 0.$$

Then it follows the general solution

$$\frac{1}{6}\xi + E = -\frac{2}{9}\log(-V) - \frac{1}{3V} + \frac{2}{9}\log(3-2V),$$

where E is a constant. Since $V(0) = -2$, then we have

$$E = -\frac{2}{9}\log 2 + \frac{1}{6} + \frac{2}{9}\log 7 = \frac{1}{6} + \frac{2}{9}\log\left(\frac{7}{2}\right).$$

Therefore, the expansion in the inner layer at $x = 1$ is $V(\xi) = H^{-1}(\xi)$ where

$$H(V) = 6\left[-\frac{2}{9}\log\left(\frac{-V}{2}\right) - \frac{1}{3V} + \frac{2}{9}\log\left(\frac{3-2V}{7}\right) - \frac{1}{6}\right].$$

Chapter 3

Method of Multiple Scales

3.1 Introductory Example

As in the previous chapter, we will introduce the ideas underlying the method by a simple example. Consider the initial value problem

$$y'' + \varepsilon y' + y = 0 \quad \text{for } t > 0 \tag{3.1.1a}$$
$$y(0) = 0, \quad y'(0) = 1 \tag{3.1.1b}$$

which models a linear oscillator with weak damping. This reduces to the linear oscillator model when $\varepsilon = 0$.

3.1.1 Regular expansion

We do not expect boundary layers since (3.1.1) is not a singular problem. This suggests that the solution might have a regular asymptotic expansion, *i.e.* we try a regular expansion

$$y(t) \sim y_0(t) + \varepsilon y_1(t) + \dots \quad \text{as } \varepsilon \longrightarrow 0 \tag{3.1.2}$$

Substituting (3.1.2) into (3.1.1) and collecting terms in equal powers of ε yields

$$\begin{aligned} y_0'' + y_0 &= 0 \\ y_n'' + y_n &= -y_{n-1}' \quad \text{for } n \geq 1, \end{aligned}$$

with initial conditions

$$y_0(0) = 0, \quad y_0'(0) = 1, \quad y_n(0) = y_n'(0) = 0, \quad n \geq 1.$$

Solving the $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$ equations we obtain

$$y(t) \sim \sin(t) - \frac{1}{2}\varepsilon t \sin(t), \tag{3.1.3}$$

but this is problematic since the correction term $y_1(t)$ contains a secular term $t\sin(t)$ which blows up as $t \longrightarrow \infty$. Consequently, the asymptotic expansion is valid for only small values of t, since $\varepsilon y_1(t) \sim y_0(t)$ when $\varepsilon t \sim 1$. The problem is that regular perturbation theory does not

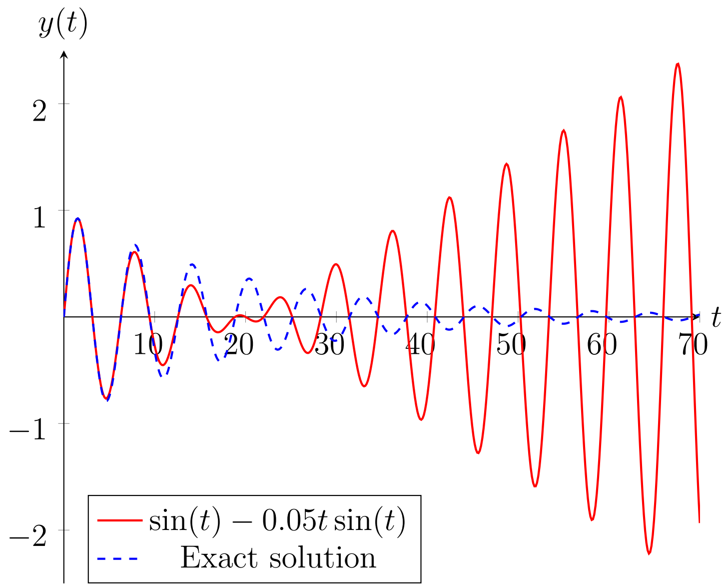


Figure 3.1: Comparison between the regular asymptotic approximation (3.1.4) and the exact solution (3.1.4) for $\varepsilon = 0.1$.

capture the correct behaviour of the exact solution. Indeed, (3.1.1) is a constant-coefficient linear ODE and it can be solved exactly:

$$y(t) = \frac{1}{\sqrt{1-\varepsilon^2/4}} e^{-\varepsilon t/2} \sin\left(t\sqrt{1-\varepsilon^2/4}\right) \tag{3.1.4}$$

It is clear that the exact solution decays but the first term in our regular asymptotic approximation (3.1.3) does not. Also, we will pick up the secular terms if we naively expand the exponential function around $t = 0$, since

$$y(t) \approx \left(1 - \frac{\varepsilon t}{2} + \frac{\varepsilon^2 t^2}{8} + \dots\right) \sin(t).$$

3.1.2 Multiple-scale expansion

In fact, there are two time-scales in the exact solution:

1. The slowly decaying exponential component which varies on a time-scale of $\mathcal{O}(1/\varepsilon)$;
2. The fast oscillating component which varies on a time-scale of $\mathcal{O}(1)$.

To identify or separate these time-scales, we introduce the variables

$$t_1 = t, \quad t_2 = \varepsilon^\alpha t, \quad \alpha > 0,$$

where t_2 is called the slow time-scale because it does not affect the asymptotic expansion until $\varepsilon^\alpha t \sim 1$. We treat these two time-scales as independent variables and consequently the original time derivative becomes

$$\frac{d}{dt} \longrightarrow \frac{dt_1}{dt}\frac{\partial}{\partial t_1} + \frac{dt_2}{dt}\frac{\partial}{\partial t_2} = \frac{\partial}{\partial t_1} + \varepsilon^\alpha \frac{\partial}{\partial t_2}. \tag{3.1.5}$$

Substituting (3.1.5) into (3.1.1) yields the transformed problem

$$\left[\partial_{t_1}^2 + 2\varepsilon^\alpha \partial_{t_1}\partial_{t_2} + \varepsilon^{2\alpha}\partial_{t_2}^2\right] y + \varepsilon\left(\partial_{t_1} + \varepsilon^\alpha \partial_{t_2}\right) y + y = 0, \tag{3.1.6a}$$

$$y(t_1,t_2)\Big|_{t_1=t_2=0} = 0, \quad \left(\partial_{t_1} + \varepsilon^\alpha\partial_{t_2}\right) y(t_1,t_2)\Big|_{t_1=t_2=0} = 1. \tag{3.1.6b}$$

Unlike the original problem, additional constraints are needed for (3.1.6) to have a unique solution, and it is precisely this degree of freedom that allows us to eliminate the secular terms!

We now introduce an asymptotic expansion

$$y \sim y_0(t_1,t_2) + \varepsilon y_1(t_1,t_2) + \dots. \tag{3.1.7}$$

Substituting (3.1.7) into (3.1.6) yields

$$\begin{aligned}\left[\partial_{t_1}^2 + 2\varepsilon^\alpha \partial_{t_1}\partial_{t_2} + \varepsilon^{2\alpha}\partial_{t_2}^2\right]&\left[y_0 + \varepsilon y_1 + \dots\right] \\ &+ \varepsilon\left(\partial_{t_1} + \varepsilon^\alpha\partial_{t_2}\right)\left(y_0 + \dots\right) + \left(y_0 + \varepsilon y_1 + \dots\right) = 0.\end{aligned}$$

The $\mathcal{O}(1)$ problem is

$$\begin{aligned}\left(\partial_{t_1}^2 + 1\right) y_0 &= 0, \\ y_0(0,0) = 0, \quad \partial_{t_1} y_0(0,0) &= 1,\end{aligned}$$

and its general solution is

$$y_0(t_1,t_2) = a_0(t_2)\sin(t_1) + b_0(t_2)\cos(t_1),$$

where $a_0(0) = 1, b_0(0) = 0$. Note that $y_0(t_1,t_2)$ consists of purely harmonic components with slowly varying amplitude. We now need to determine α in the slow time-scale t_2. Observe that for $\alpha > 1$ the $\mathcal{O}(\varepsilon)$ equation is

$$\left(\partial_{t_1}^2 + 1\right) y_1 = -\partial_{t_1} y_0,$$

and the inhomogeneous term $\partial_{t_1} y_0$ will generate secular terms, since it belongs to the kernel of homogeneous linear operator $\left(\partial_{t_1}^2 + 1\right)$. More importantly, there is no way to generate non-trivial solution that will cancel the secular term. This can be prevented by choosing $\alpha = 1$.

The $\mathcal{O}(\varepsilon)$ equation is

$$\begin{aligned}\left(\partial_{t_1}^2 + 1\right) y_1 &= -2\partial_{t_1}\partial_{t_2} y_0 - \partial_{t_1} y_0, \\ y_1(0,0) = 0, \quad \partial_{t_1} y_1(0,0) &+ \partial_{t_2} y_0(0,0) = 0.\end{aligned}$$

Substituting y_0 gives

$$\begin{aligned}\left(\partial_{t_1}^2 + 1\right) y_1 &= -2\left(a_0'\cos(t_1) - b_0'\sin(t_1)\right) - \left(a_0\cos(t_1) - b_0\sin(t_1)\right) \\ &= \left(2b_0' + b_0\right)\sin(t_1) - \left(2a_0' + a_0\right)\cos(t_1).\end{aligned}$$

The general solution of the $\mathcal{O}(\varepsilon)$ problem is

$$\begin{aligned}y_1(t_1,t_2) &= a_1(t_2)\sin(t_1) + b_1(t_2)\cos(t_1) \\ &\quad - \frac{1}{2}\left(2b_0' + b_0\right)\underbrace{t_1\cos(t_1)}_{\text{secular}} - \frac{1}{2}\left(2a_0' + a_0\right)\underbrace{t_1\sin(t_1)}_{\text{secular}},\end{aligned}$$

with $a_1(0) = b_0'(0), b_1(0) = 0$. We can choose the functions a_0, b_0 to remove the secular terms, which results in

$$2b_0' + b_0 = 0 \implies b_0(t_2) = \beta_0 e^{-t_2/2} = 0, \quad \text{since } b_0(0) = 0,$$

and

$$2a_0' + a_0 = 0 \implies a_0(t_2) = \alpha_0 e^{-t_2/2} = e^{-t_2/2}, \quad \text{since } a_0(0) = 0.$$

Hence, a first term approximation of the solution $y(t)$ of (3.1.1) is

$$y \sim e^{-\varepsilon t/2}\sin(t).$$

One can prove that this asymptotic expansion is uniformly valid for $0 \leq t \leq \mathcal{O}(1/\varepsilon)$.

3.1.3 Discussion

1. Many problems have the $\mathcal{O}(1)$ equation as

$$y_0'' + \omega^2 y_0 = 0.$$

   and the general solution is

$$y_0(t) = a\cos(\omega t) + b\sin(\omega t).$$

   If the original problem is nonlinear and the $\mathcal{O}(1)$ equation is as above, then it is usually more convenient to use a complex representation of y_0, *i.e.*

$$y(t) = Ae^{i\omega t} + \bar{A}e^{-i\omega t} = B\cos(\omega t + \theta).$$

   These complex representations make identify the secular terms much easier.

2. Often, higher-order equations have the form

$$y_n'' + \omega^2 y_n = f(t).$$

   A secular term arises if $f(t)$ contains a solution of the $\mathcal{O}(1)$ problem, *e.g.* $\cos(\omega t)$ or $\sin(\omega t)$. We can avoid secular terms by requiring the t_2-dependent coefficients of $\cos(\omega t_1)$ and $\sin(\omega t_1)$ to vanish. For example, there are no secular terms if

$$f(t) = \sin(\omega t)\cos(\omega t) = \sin(2\omega t)/2,$$

   but there is a secular term if

$$f(t) = \cos^3(\omega t) = \frac{1}{4}\left(3\cos(\omega t) + \cos(3\omega t)\right).$$

3. The time scales should be modified depending on the problem. Some possibilities include:

   (a) Several time-scales: *e.g.* $t_1 = t/\varepsilon, t_2 = t, t_3 = \varepsilon t, \ldots$.

(b) More complex ε-dependency:

$$t_1 = \underbrace{\left(1 + \omega_1\varepsilon + \omega_2\varepsilon^2 + \dots\right)}_{\text{expansion of the effective frequency}} t, \quad t_2 = \varepsilon t.$$

This is called the *Lindstedt's method* or the *method of strained coordinates*.

(c) Correct scaling may not be obvious, so we might start off with

$$t_1 = \varepsilon^\alpha t, \quad t_2 = \varepsilon^\beta t, \quad \alpha < \beta.$$

(d) Nonlinear time-dependence:

$$t_1 = f(t, \varepsilon), \quad t_2 = \varepsilon t.$$

3.2 Forced Motion Near Resonance

In this section, we consider an extension of the introductory example: a dampled nonlinear oscillator that is forced at a frequency near resonance. As an example, we will study the damped Duffing equation

$$y'' + \varepsilon\lambda y' + y + \varepsilon\kappa y^3 = \varepsilon\cos\Big((1 + \varepsilon\omega)t\Big) \quad \text{for } t > 0 \tag{3.2.1a}$$

$$y(0) = 0, \quad y'(0) = 0. \tag{3.2.1b}$$

The damping term $\varepsilon\lambda y'$, nonlinear correction term $\varepsilon\kappa y^3$ and forcing term $\varepsilon\cos\Big((1+\varepsilon\omega)t\Big)$ are small. Also, ω, λ, κ are constants with λ and κ nonnegative. We expect the solution to be small due to the small forcing and zero initial conditions.

Consider the simpler equation

$$y'' + y = \varepsilon\cos(\Omega t), \quad \Omega \neq \pm 1, \quad y(0) = y'(0) = 0. \tag{3.2.2}$$

The unique solution is

$$y(t) = \frac{\varepsilon}{1 - \Omega^2}\Big[\cos(\Omega t) - \cos(t)\Big] \tag{3.2.3}$$

and the solution blows up as expected when the driving frequency $\Omega \approx 1$. To understand the situation, suppose $\Omega = 1 + \varepsilon\omega$. The particular solution of (3.2.2) is given by

$$y_p(t) = \begin{cases} -\dfrac{1}{\omega(2 + \varepsilon\omega)}\cos\Big((1 + \varepsilon\omega)t\Big) & \text{if } \omega \neq 0, -2/\varepsilon, \\ \dfrac{1}{2}\varepsilon t\sin(t) & \text{otherwise.} \end{cases} \tag{3.2.4}$$

In both cases a relatively small, order $\mathcal{O}(\varepsilon)$, forcing results in at least an $\mathcal{O}(1)$ solution. Moreover, the behaviour of the solution depends on ω, which is typical of a forcing system.

We take $t_1 = t$ and $t_2 = \varepsilon t$, although we should take $t_2 = \varepsilon^\alpha t, \alpha > 0$ in general to allow for some flexibility. The forced Duffing equation becomes

$$\Big[\partial_{t_1}^2 + 2\varepsilon\partial_{t_1}\partial_{t_2} + \varepsilon^2\partial_{t_2}^2\Big]y + \varepsilon\lambda\Big[\partial_{t_1} + \varepsilon\partial_{t_2}\Big]y + y + \varepsilon\kappa y^3 = \varepsilon\cos\left(t_1 + \varepsilon\omega t_1\right). \tag{3.2.5}$$

Although we expect the leading-order term in the expansion to be $\mathcal{O}(\varepsilon)$, the solution can become larger near a resonant frequency. Because it is not clear what amplitude the solution actually reaches, we guess a general asymptotic expansion of the form

$$y \sim \varepsilon^\beta y_0(t_1, t_2) + \varepsilon^\gamma y_1(t_1, t_2) + \ldots, \quad \beta < \gamma. \tag{3.2.6}$$

We also assume that $\beta < 1$ due to the resonance effect. Substituting (3.2.6) into (3.2.5) gives

$$\begin{aligned}
&\Big[\varepsilon^\beta \partial_{t_1}^2 y_0 + \underbrace{2\varepsilon^{1+\beta}\partial_{t_1}\partial_{t_2} y_0}_{④} + \underbrace{\varepsilon^\gamma \partial_{t_1}^2 y_1}_{①} + \ldots\Big] + \Big[\underbrace{\varepsilon^{1+\beta}\lambda\partial_{t_1} y_0}_{②} + \ldots\Big] \\
&\quad + \Big[\varepsilon^\beta y_0 + \underbrace{\varepsilon^\gamma y_1}_{①} + \ldots\Big] + \Big[\underbrace{\varepsilon^{1+3\beta}\kappa y_0^3}_{②} + \ldots\Big] = \underbrace{\varepsilon \cos\left(t_1 + \varepsilon w t_1\right)}_{③}.
\end{aligned}$$

The $\mathcal{O}(\varepsilon^\beta)$ problem is

$$\begin{aligned}
\left(\partial_{t_1}^2 + 1\right) y_0 &= 0, \\
y_0(0,0) = \partial_{t_1} y_0(0,0) &= 0,
\end{aligned}$$

and its general solution is

$$y_0 = A(t_2)\cos(t_1 + \theta(t_2)),$$

with $A(0) = 0$.

We need to determine β and γ before proceed any further. The terms ② concern with the preceeding solution y_0 and the term ③ is the forcing term. For the most complete approximation, the problem for the second term y_1 in the expansion (3.2.6), which comes from the terms ①, must deal with both ② and ③. This is possible if we choose $\gamma = 1$ and $\beta = 0$. The $\mathcal{O}(\varepsilon)$ equation in

$$\begin{aligned}
\left(\partial_{t_1}^2 + 1\right) y_1 &= -2\partial_{t_1}\partial_{t_2} y_0 - \lambda\partial_{t_1} y_0 - \kappa y_0^3 + \cos(t_1 + \omega t_2) \\
&= \Big[2A' + \lambda A\Big]\sin(t_1 + \theta) + 2\theta' A\cos(t_1 + \theta) \\
&\quad - \frac{\kappa}{4}A^3\Big[3\cos(t_1 + \theta) + \cos\Big(3(t_1 + \theta)\Big)\Big] + \cos(t_1 + \omega t_2).
\end{aligned}$$

Note that

$$\begin{aligned}
\cos(t_1 + \omega t_2) &= \cos(t_1 + \theta - \theta + \omega t_2) \\
&= \cos(t_1 + \theta)\cos(\theta - \omega t_2) + \sin(t_1 + \theta)\sin(\theta - \omega t_2).
\end{aligned}$$

Thus, we can remove the secular terms $\sin(t_1 + \theta)$ and $\cos(t_1 + \theta)$ by requiring

$$2A' + \lambda A = -\sin(\theta - \omega t_2) \tag{3.2.7a}$$

$$2\theta' A - \frac{3\kappa}{4}A^3 = -\cos(\theta - \omega t_2). \tag{3.2.7b}$$

From $A(0) = 0$ and assuming $A'(0) > 0$, it follows that $\theta(0) = -\pi/2$.

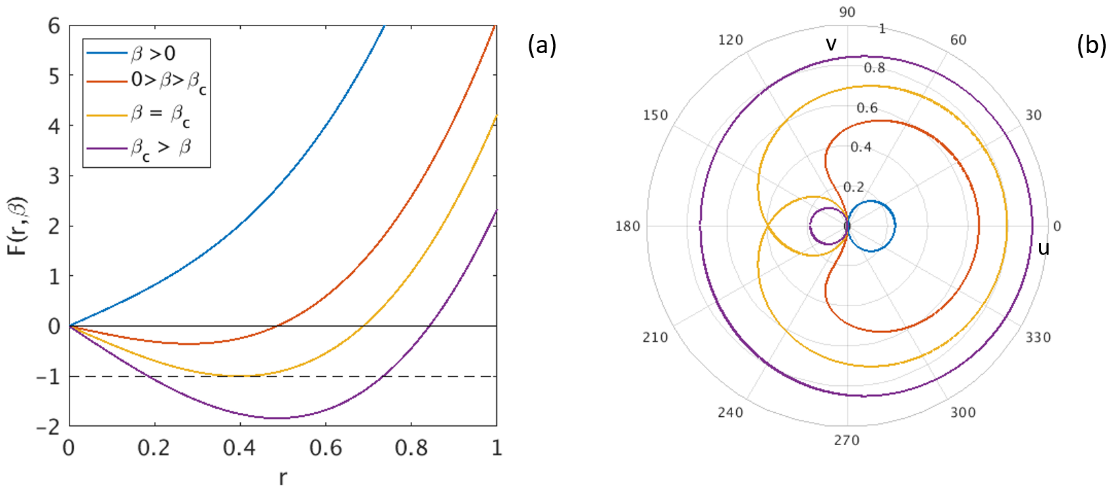


Figure 3.2: Nullcline for ϕ_τ. (a) $F(r,\beta)$ as a function of r with varying β. (b) Nullcline for ϕ_τ with varying β. Parameter are given by $\gamma = 0.75$ and $\beta = -\beta_c, \beta_c/2, \beta_c, 1.5\beta_c$, respectively.

It remains to solve (3.2.7) with initial conditions $A(0) = 0, \theta(0) = -\pi/2$ to find the amplitude function $A(t_2)$ and phase function $\theta(t_2)$. For the analytic simplicity, changing variables as $r = \sqrt{\kappa}A/2$ and $\phi = \theta - wt_2$ gives

$$\begin{cases} 2r' = -\lambda r - \frac{\gamma}{2}\sin\phi, \\ 2\phi' = \beta + 3r^2 - \frac{\gamma}{2r}\cos\phi. \end{cases} \tag{3.2.8}$$

where $\gamma = \sqrt{\kappa}$ and $\beta = -2\omega$. We now analyze the rewritten amplitude equation (3.2.8). The nullcline for r_τ is $r = -\gamma\sin\theta/2\lambda$. Similarly, nullcline for ϕ_τ is given by $\cos\theta = 2r(\beta+3r^2)/\gamma \equiv F(r,\beta)$, see Fig. 3.2:

- If $\beta > 0$, there is unique r for each θ, see the blue line.
- If $0 > \beta > \beta_c$ where $\min_r F(r,\beta_c) = -1$ (and it turns out that $\beta_c^3 = -81\gamma^2/16$), then there are two values of r for each $\cos\theta$ in some interval $(-z, 0)$ for some $z \in [0,1]$. See the red line.
- If $\beta < \beta_c$, then two values of r exist for all $\cos\theta$ between -1 and 0. See the purple line.

For $0 > \beta > \beta_c$, then the non-trivial fixed point (FB) stability of (3.2.8) with varying the nullcline r_τ for $\lambda \geq 0$ is the following, see Fig. 3.3:

- For small λ, only one stable fixed point, see the curve A intersecting with the red line.

B,C If $\lambda = \lambda_{1C}$, there is a SN bifurcation, that is, saddle and a stable FP. See the curve B and B intersecting with the red line.

- At $\lambda = \lambda_{2C}$, there is a second SN bifurcation in which saddle and other stable FP (from A) annihilate leaning the stable FP (from B). See the curve D and E intersecting with the red line.

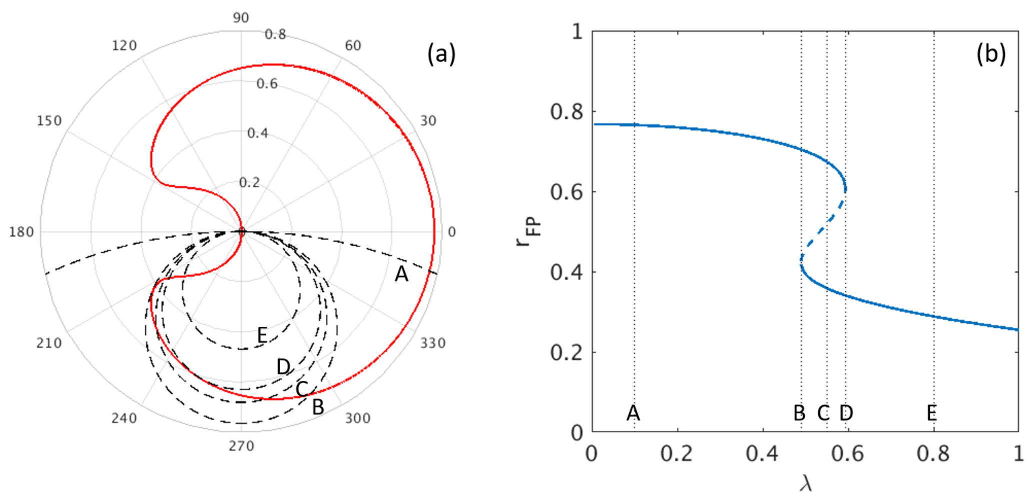


Figure 3.3: Non-trivial fixed points as a function of λ and its bifurcation diagram. (a) Intersections of nullcline ϕ_τ and r_τ with varying λ. (b) Bifurcation diagram of fixed radius r_{FP} as a function of λ.

3.3 Periodically Forced Nonlinear Oscillators

This section is taken from [Bre14, Chapter 1.2] and [PRK03, Chapter 7.1]. Consider a general model of a nonlinear oscillator

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{f}(\boldsymbol{u}), \quad \boldsymbol{u} = (u_1, \dots, u_M), \quad \text{with } M \geq 2. \tag{3.3.1}$$

For example, u_1 might represents the membrane potential of the neuron (treated as a point processor) and $u_2, \dots, \boldsymbol{u}_M$ represent various ionic channel gating variables. Suppose there exists a stable periodic solution $\boldsymbol{U}(t) = \boldsymbol{U}(t+\Delta_0)$, where $\omega_0 = 2\pi/\Delta_0$ is the natural frequency of the oscillator. In *phase space*, the solution is an isolated attractive trajectory called a *limit cycle*. The dynamics on the limit cycle can be described by a uniformly rotating phase, *i.e.*

$$\frac{d\phi}{dt} = \omega_0 \quad \text{and} \quad \boldsymbol{U}(t) = \boldsymbol{g}(\phi(t)), \tag{3.3.2}$$

with $\boldsymbol{g}$ a 2π-periodic function. The phase ϕ should be viewed as a coordinate along the limit cycle, such that it grows monotonically in the direction of the motion and gains 2π during each rotation. Note that the phase is *neutrally stable* with respect to perturbations along the limit cycle - this reflects the time-shift invariance of an autonomous dynamical system. *On the limit cycle, the time shift Δt is equivalent to the phase shift $\Delta\phi = \omega_0\Delta t$.* Now, suppose that a small external periodic input is applied to the oscillator such that

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{f}(\boldsymbol{u}) + \varepsilon \boldsymbol{P}(\boldsymbol{u}, t), \tag{3.3.3}$$

where $\boldsymbol{P}(\boldsymbol{u}, t) = \boldsymbol{P}(\boldsymbol{u}, t + \Delta)$ with $\omega = 2\pi/\Delta$ the forcing frequency. If the amplitude ε is sufficiently small and the cycle is stable, then the resulting deviations transverse to the limit cycle are small so that the main effect of the perturbation is a phase-shift along the limit cycle. This suggests a description of the perturbed dynamics with the phase variable only. Therefore, we need to extend the definition of phase to a neighbourhood of the limit cycle.

3.3.1 Isochrones

Roughly speaking, the idea is to define the phase variable in such a way that it rotates uniformly on the limit cycle as well as its neighbourhood. Suppose that we observe the unperturbed system stroboscopically at time intervals of length Δ_0. This leads to a Poincaré mapping

$$\boldsymbol{u}(t) \longrightarrow \boldsymbol{u}(t+\Delta_0) \equiv G(\boldsymbol{u}(t)).$$

The map G has all points on the limit cycle as fixed points. Choose a point $\boldsymbol{U}^*$ on the limit cycle and consider all points in a neighbourhood of $\boldsymbol{U}^*$ in $\mathbb{R}^M$ that are attracted to it under the action of Φ. They form an $(M-1)$-dimensional hypersurface I, called an **isochrone**, crossing the limit cycle at $\boldsymbol{U}^*$. A unique isochrone can be drawn through each point on the limit cycle so we can parameterise the isochrones by the phase ϕ, *i.e.* $I = I(\phi)$. Finally, we extend the definition of phase to the vicinity of the limit cycle by taking all points $\boldsymbol{u} \in I(\phi)$ to have the same phase, $\Phi(\boldsymbol{u}) = \phi$, which then rotates at the natural frequency ω_0 (in the unperturbed case).

Example 3.3.1. Consider the following complex amplitude equation that arises for a limit cycle oscillator close to a Hopf bifurcation:

$$\frac{dA}{dt} = (1+i\eta)A - (1+i\alpha)|A|^2, \quad A \in \mathbb{C}.$$

In polar coordinates $A = Re^{i\theta}$, we have

$$\begin{aligned} \frac{dR}{dt} &= R(1-R^2) \\ \frac{d\theta}{dt} &= \eta - \alpha R^2. \end{aligned}$$

Observe that the origin is unstable and the unit circle is a stable limit cycle. The solution for arbitrary initial data $R(0) = R_0$, $\theta(0) = \theta_0$ is

$$\begin{aligned} R(t) &= \left[1 + \left(\frac{1-R_0^2}{R_0}\right) e^{-2t}\right]^{-1/2} \\ \theta(t) &= \theta_0 + \omega_0 t - \frac{\alpha}{2} \ln\left[R_0^2 + (1-R_0^2)e^{-2t}\right], \end{aligned}$$

where $\omega_0 = \eta - \alpha$ is the natural frequency of the stable limit cycle at $R = 1$. Strobing the solution at time $t = n\Delta_0$, we see that

$$\lim_{n\to\infty} \theta(n\Delta_0) = \theta_0 - \alpha \ln R_0.$$

Hence, we can define a phase on the whole plane as

$$\Phi(R,\theta) = \theta - \alpha \ln R$$

and the isochrones are the lines of constant phase Φ, which are logarithmic spirals on the (R,θ) plane. We verify that this phase rotates uniformly:

$$\frac{d\Phi}{dt} = \frac{d\theta}{dt} - \frac{\alpha}{R}\frac{dR}{dt} = \eta - \alpha R^2 - \alpha(1-R^2) = \eta - \alpha = \omega_0.$$

It seems like the angle variable θ can be taken to be the phase variable Φ since it rotates with a constant angular velocity ω_0. However, if the initial amplitude deviates from unity, an additional phase shift occurs due to the term proportional to α in the $\dot{\theta}$-equation. It can be seen from $\theta(t)$ and $R(t)$ that the additional phase shift is $-\alpha \ln R_0$.

3.3.2 Phase equation

For an unperturbed oscillator in the vicinity of the limit cycle, we have from (3.3.1) and (3.3.2)

$$\omega_0 = \frac{d\Phi(\boldsymbol{u})}{dt} = \sum_{k=1}^{M} \frac{\partial \Phi}{\partial u_k} \frac{du_k}{dt} = \sum_{k=1}^{M} \frac{\partial \Phi}{\partial u_k} f_k(\boldsymbol{u}).$$

Now consider the perturbed system (3.3.3) but with the "'unperturbed" definition of the phase:

$$\frac{d\Phi(\boldsymbol{u})}{dt} = \sum_{k=1}^{M} \frac{\partial \Phi}{\partial u_k} \Big(f_k(\boldsymbol{u}) + \varepsilon P_k(\boldsymbol{u}, t) \Big) = \omega_0 + \varepsilon \sum_{k=1}^{M} \frac{\partial \Phi}{\partial u_k} P_k(\boldsymbol{u}, t).$$

Because the sum is $\mathcal{O}(\varepsilon)$ and the deviations of $\boldsymbol{u}$ from the limit cycle $\boldsymbol{U}$ are small, to a first approximation, we can neglect these deviations and calculate the sum on the limit cycle. Consequently,

$$\frac{d\Phi(\boldsymbol{u})}{dt} = \omega_0 + \varepsilon \sum_{k=1}^{M} \frac{\partial \Phi(\boldsymbol{U})}{\partial u_k} P_k(\boldsymbol{U}, t).$$

Finally, since points on the limit cycle are in one-to-one correspondence with the phase θ, we obtain the closed phase equation

$$\frac{d\phi}{dt} = \omega_0 + \varepsilon Q(\phi, t), \tag{3.3.4}$$

where

$$Q(\phi, t) = \sum_{k=1}^{M} \frac{\partial \Phi(\boldsymbol{U}(\phi))}{\partial u_k} P_k(\boldsymbol{U}(\phi), t) \tag{3.3.5}$$

is a 2π-periodic function of ϕ and a Δ-periodic function of t. The phase equation (3.3.4) describes the dynamics of the phase of a periodic oscillator in the presence of a small periodic external force and $Q(\phi, t)$ contains all the information of the dynamical system. This is known as the **phase reduction method**.

Example 3.3.2. Returning to Example 3.3.1, the system in Cartesian coordinate is

$$\begin{aligned}
\frac{dx}{dt} &= x - \eta y - \left(x^2 + y^2\right)(x - \eta y) + \varepsilon \cos(\omega t) \\
\frac{dy}{dt} &= y + \eta y - \left(x^2 + y^2\right)(y + \alpha x)
\end{aligned}$$

where we periodically force the nonlinear oscillator in the x-direction. The isochrone is given by

$$\Phi = \arctan\left(\frac{y}{x}\right) - \frac{\alpha}{2} \ln\big(x^2 + y^2\big),$$

and differentiating with respect to x yields

$$\frac{\partial \Phi}{\partial x} = -\frac{y}{x^2 + y^2} - \frac{\alpha x}{x^2 + y^2}.$$

On the limit cycle $\boldsymbol{u}_0 = (x_0, y_0) = (\cos\phi, \sin\phi)$, we have

$$\frac{\partial \Phi}{\partial x}\left(\boldsymbol{u}_0(\phi)\right) = -\sin\phi - \alpha\cos\phi.$$

It follows that the corresponding phase equation is

$$\frac{d\phi}{dt} = \omega_0 - \varepsilon\left(\alpha\cos\phi + \sin\phi\right)\cos(\omega t).$$

3.3.3 Phase resetting curves

In neuroscience, the function $Q(\phi, t)$ can be related to an easily measurable property of a neural oscillator, namely its **phase resetting curves (PRC)**. Let us denote this by a 2π-periodic function $R(\phi)$. For a neural oscillator, the PRC is found experimentally by perturbing the oscillator with an impulse at different times in its cycle and measuring the resulting phase shift from the unperturbed oscillator. Suppose we perturb u_1, it follows from (3.3.4) that

$$\frac{d\phi}{dt} = \omega_0 + \varepsilon\left(\frac{\partial \Phi(\boldsymbol{U}(\phi))}{\partial u_1}\right)\delta(t - t_0).$$

Integrating over a small interval around t_0, we see that the impulse induces a phase shift $\Delta\phi = \varepsilon R(\phi_0)$, where

$$R(\phi) = \frac{\partial \Phi(\boldsymbol{U}(\phi))}{\partial u_1} \quad \text{and} \quad \phi_0 = \phi(t_0).$$

Given the phase resetting curve $R(\phi)$, a general time-dependent voltage perturbation $\varepsilon P(t)$ is determined by the phase equation

$$\frac{d\phi}{dt} = \omega_0 + \varepsilon R(\phi)P(t) = \omega_0 + \varepsilon Q(\phi, t).$$

We can also express the PRC in terms of the firing times of a neuron. Let T^n be the nth firing time of the neuron. Consider the phase $\phi = 0$. In the absence of perturbation, we have $\phi(t) = 2\pi t/\Delta_0$ so the firing times are $T^n = n\Delta_0$. On the other hand, a small perturbation applied at the point ϕ on the limit cycle at time $t \in (T^n, T^{n+1})$, induces a phase shift that changes the next firing time. Depending on the type of neurons, the impulse either advance or delay the onset of the next spike. Oscillators with a strictly positive PRC $R(\phi)$ are called type I, whereas those for which the PRC has a negative regime are called type II.

3.3.4 Averaging theory

In the zero-order approximation, *i.e.* $\varepsilon = 0$, the phase equation (3.3.4) gives rise to $\phi(t) = \phi_0 + \omega_0 t$. Since $Q(\phi, t)$ is 2π-periodic in ϕ and Δ-periodic in t, we expand $Q(\phi, t)$ as a double Fourier series

$$\begin{aligned} Q(\phi, t) &= \sum_{l,k} a_{l,k} e^{ik\phi + il\omega t} \\ &= \sum_{l,k} a_{l,k} e^{ik\phi_0} e^{i(k\omega_0 + l\omega)t}, \end{aligned}$$

where $\omega = 2\pi/\Delta$. Thus Q contains fast oscillating terms (compared to the time scale $1/\varepsilon$) together with slowly varying terms, the latter satisfy the *resonance condition*

$$k\omega_0 + l\omega \approx 0.$$

Substituting this double Fourier series into the phase equation (3.3.4), we see that the fast oscillating terms lead to $\mathcal{O}(\varepsilon)$ phase deviations, while the resonant terms can lead to large variations of the phase and are mostly important for the dynamics. Thus we have to average the forcing term Q keeping only the resonant terms. We now identify the resonant terms using the resonance condition above:

1. The simplest case is $\omega \approx \omega_0$ for which the resonant terms satisfy $l = -k$. This results in an averaged forcing
$$Q(\phi, t) \approx \sum_k a_{-k,k} e^{ik(\phi - \omega t)} = q(\phi - \omega t)$$
and the phase equation becomes
$$\frac{d\phi}{dt} = \omega_0 + \varepsilon q(\phi - \omega t).$$
Introducing the phase difference $\psi = \phi - \omega t$ between the oscillator and external input, we obtain
$$\frac{d\psi}{dt} = -\Delta\omega + \varepsilon q(\psi),$$
where $\Delta\omega = \omega - \omega_0$ is the degree of **frequency detuning**.

2. The other case is $\omega \approx m\omega_0/n$, where m and n are coprime. The forcing term becomes
$$Q(\phi, t) \approx \sum_k a_{-nk,mk} e^{ik(m\phi - n\omega t)} = \widehat{q}(m\phi - n\omega t)$$
and the phase equation has the form
$$\frac{d\phi}{dt} = \omega_0 + \varepsilon \widehat{q}(m\phi - n\omega t).$$
Introducing the phase difference $\psi = m\phi - n\omega t$, we obtain
$$\frac{d\psi}{dt} = m\omega_0 - n\omega + \varepsilon m \widehat{q}(\psi),$$
where the frequency detuning is $\Delta\omega = n\omega - n\omega_0$ instead.

The above analysis is an application of the averaging theorem. Assuming $\Delta\omega = \omega - \omega_0 = \mathcal{O}(\varepsilon)$ and setting $\psi = \phi - \omega t$, we have

$$\frac{d\psi}{dt} = -\Delta\omega + \varepsilon Q(\psi + \omega t, t) = \mathcal{O}(\varepsilon).$$

Define

$$q(\psi) = \lim_{T \longrightarrow \infty} \frac{1}{T} \int_0^T Q(\psi + \omega t, t)\, dt,$$

and consider the averaged equation

$$\frac{d\bar{\psi}}{dt} = -\Delta\omega + \varepsilon q(\bar{\psi}),$$

where q only contains the resonant terms of Q as above. The averaging theorem guarantees that there exists a change of variable $\psi = \bar{\psi} + \varepsilon w(\varphi, t)$ that maps solutions of the full equation to those of the averaged equation to leading order in ε. In general, one can only establish that a solution of the full equation is ε-close to a corresponding solution of the average equation for times of $\mathcal{O}(1/\varepsilon)$, *i.e.*

$$\sup_{t\in I} \left|\psi(t) - \bar{\psi}(t)\right| \leq C\varepsilon.$$

3.3.5 Phase-locking and synchronisation

We now discuss the solutions of the averaged phase equation

$$\frac{d\psi}{dt} = -\Delta\omega + \varepsilon q(\psi). \tag{3.3.6}$$

Suppose that the 2π-periodic function $q(\psi)$ has a unique maximum $q_{\max}$ and a unique minimum $q_{\min}$. The fixed points ψ^* of (3.3.6) satisfy $\varepsilon q(\psi^*) = \Delta\omega$.

1. **Synchronisation regime**
   If the degree of detuning is sufficiently small, in the sense that

   $$\varepsilon q_{\min} < \Delta\omega < \varepsilon q_{\max},$$

   then there exists at least one pair of stable/unstable fixed points (ψ_s, ψ_u). (This follows from the fact that $q(\psi)$ is 2π-periodic and continuous so it has to cross any horizontal line an even number of times.) The system then evolves to the solution $\phi(t) = \omega t + \psi_s$ and this is the *phase-locked synchronise state*. The oscillator is also said to be *frequency entrained*, meaning that the frequency of the oscillator coincides with that of the external force.

2. **Drift regime**
   Increasing $|\Delta\omega|$ means that ψ_s, ψ_u coalesce at a saddle point, beyond which there are no fixed points. This results in a saddle-node bifurcation and phase-locking disappears. If $|\Delta\omega$ is large, then $\dot{\psi}$ never changes sign and the oscillation frequency differs from the forcing frequency. The phase $\psi(t)$ rotates through 2π with period

   $$T_\psi = \left|\int_0^{2\pi} \frac{d\psi}{\varepsilon q(\psi) - \Delta\omega}\right|.$$

   The mean frequency of rotation is thus $\Omega = \omega + \Omega_\psi$, where $\Omega_\psi = 2\pi/T_\psi$ is the *beat frequency*.

   For a fixed ε, suppose that $\Delta\omega$ is close to one of the bifurcation point $\Delta\omega_{\max} := \varepsilon q_{\max}$. The integral in T_ψ is dominated by a small region around $\psi_{\max}$ and expanding $q(\psi)$ around $\psi_{\max}$ yields

   $$\begin{aligned}\Omega_\psi &\approx 2\pi\left|\int_{-\infty}^{\infty} \frac{d\psi}{\varepsilon q''(\psi_{\max})\psi^2 - (\Delta\omega - \Delta\omega_{\max})}\right|^{-1} \\ &= \sqrt{\varepsilon|q''(\psi_{\max})|(\Delta\omega - \Delta\omega_{\max})} = \mathcal{O}(\sqrt{\varepsilon})\end{aligned}$$

3.3.6 Phase reduction for networks of coupled oscillators

We extend the analysis to a network of N coupled oscillators. Let $\boldsymbol{u}_i \in \mathbb{R}^M, i = 1, \ldots, N$ denote the state of the ith oscillator. The general model can be written as

$$\frac{d\boldsymbol{u}_i}{dt} = \boldsymbol{f}(\boldsymbol{u}_i) + \varepsilon \sum_{j=1}^{N} a_{ij} \boldsymbol{H}(\boldsymbol{u}_j), \quad i = 1, \ldots, N, \tag{3.3.7}$$

where the first term represents the local autonomous dynamics and the second term describes the interaction between oscillators. In a similar fashion to a single periodically forced oscillator, we can write down the phase equation:

$$\frac{d\phi_i(\boldsymbol{u}_i)}{dt} = \omega_0 + \varepsilon \left(\frac{\partial \phi_i}{\partial \boldsymbol{u}_i}\right) \cdot \left(\sum_{j=1}^{N} a_{ij} \boldsymbol{H}(\boldsymbol{u}_j)\right), \quad i = 1, \ldots, N. \tag{3.3.8}$$

Since the limit cycle is uniquely defined by phase,

$$\frac{d\phi_i}{dt} = \omega_0 + \varepsilon \sum_{j=1}^{N} a_{ij} Q_i(\phi_i, \phi_j), \quad i = 1, \ldots, N, \tag{3.3.9}$$

where

$$Q_i(\phi_i, \phi_j) = \frac{\partial \phi_i}{\partial \boldsymbol{u}_i}(\boldsymbol{U}(\phi_i)) \cdot \boldsymbol{H}(\boldsymbol{U}(\phi_j)). \tag{3.3.10}$$

The final step is to use the method of averaging to obtain the phase-difference equation. Introducing $\psi_i = \phi_i - \omega_0 t$, we obtain

$$\frac{d\psi_i}{dt} = \varepsilon \sum_{j=1}^{N} a_{ij} Q_i\left(\psi_i + \omega_0 t, \psi_j + \omega_0 t\right).$$

Upon averaging over one period, we obtain

$$\frac{d\psi_i}{dt} = \varepsilon \sum_{j=1}^{N} a_{ij} h(\psi_j - \psi_i), \tag{3.3.11}$$

where

$$\begin{aligned} h(\psi_j - \psi_i) &= \frac{1}{\Delta_0} \int_0^{\Delta_0} \boldsymbol{R}(\psi_i + \omega_0 t) \cdot \left(\sum_{j=1}^{N} \boldsymbol{H}(\boldsymbol{U}(\psi_j + \omega_0 t))\right) dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \boldsymbol{R}(\phi + \psi_i - \psi_j) \cdot \left(\sum_{j=1}^{N} \boldsymbol{H}(\boldsymbol{U}(\phi))\right) d\phi, \end{aligned}$$

with $\phi = \psi_j + \omega_0 t$. Here, $\boldsymbol{R}$ is the phase resetting curve.

Phase-locked solutions

We define a one-to-one phase-locked solutions to be

$$\psi_i(t) = t\Delta w + \overline{\psi}_i, \tag{3.3.12}$$

where $\overline{\psi}_i$ is constant. Taking time derivative on (3.3.12) and imposing (3.3.11) yields

$$\Delta\omega = \varepsilon \sum_{j=1}^{N} a_{ij} h\left(\overline{\psi}_j - \overline{\psi}_i\right), \quad i = 1, \ldots, N. \tag{3.3.13}$$

Since we have N equations in N unknowns $\Delta\omega$ and $N-1$ phases $\overline{\psi}_j - \overline{\psi}_1$, then one can find the phase-locked solutions (We only care about phase difference.)

Stability

In order to determine local stability, we set

$$\psi_i(t) = \overline{\psi}_i + t\Delta\omega + \Delta\psi_i(t). \tag{3.3.14}$$

To linearize it, taking time derivative on (3.3.14) and imposing the phase-locked solutions (3.3.13) gives

$$\frac{d\Delta\psi_i}{dt} = \varepsilon \sum_{j=1}^{N} \widehat{H}_{ij}\left(\Phi\right)\Delta\psi_j, \tag{3.3.15}$$

where $\Phi = \left(\bar{\psi}_1, \ldots \bar{\psi}_N\right)$ and

$$\widehat{H}_{ij}\left(\Phi\right) = a_{ij} h\left(\overline{\psi}_j - \overline{\psi}_i\right) - \delta_{ij} \sum_{k} a_{ik} h\left(\overline{\psi}_k - \overline{\psi}_i\right). \tag{3.3.16}$$

Pair of identical oscillators

For example, we assume that $N = 2$ and symmetric coupling, that is $a_{12} = a_{21}$ and $a_{11} = a_{22} = 0$ (no self-interaction). Let $\psi = \psi_2 - \psi_1$. Then (3.3.11) turns out to be

$$\frac{d\psi}{dt} = \varepsilon H^{-}(\psi),$$

where $H^{-1}(\psi) = h(-\psi) - h(\psi)$. Imposing the assumption on (3.3.13) implies that the phase-locked states are given by zeros of the odd function, $H^{-}(\psi) = 0$. Furthermore, it is stable if

$$\varepsilon \frac{dH^{-}(\psi)}{d\psi} < 0.$$

By symmetry and periodicity, the in-phase solution $\psi = 0$ and anti-phase solution $\psi = \pi$ are guaranteed to exist.

3.4 Partial Differential Equations

In this section, we apply the method of multiple scales to the linear wave equation and the nonlinear Klein-Gordon equation.

3.4.1 Elastic string with weak damping

Consider the one-dimensional wave equation with weak damping:

$$\partial_x^2 u = \partial_t^2 u + \varepsilon \partial_t u, \quad 0 < x < 1, \quad t > 0, \tag{3.4.1a}$$
$$u = 0 \quad \text{at } x = 0 \text{ and } x = 1, \tag{3.4.1b}$$
$$u(x,0) = g(x), \quad \partial_t u(x,0) = 0. \tag{3.4.1c}$$

Similar to the weakly damped oscillator, we introduce two separate time scales $t_1 = t$, $t_2 = \varepsilon t$. In this case, (3.4.1) becomes

$$\partial_x^2 u = \left[\partial_{t_1}^2 + 2\varepsilon \partial_{t_1}\partial_{t_2} + \varepsilon^2 \partial_{t_2}^2\right] u + \varepsilon \left[\partial_{t_1} + \varepsilon \partial_{t_2}\right] u, \tag{3.4.2a}$$
$$u = 0 \quad \text{at } x = 0 \text{ and } x = 1, \tag{3.4.2b}$$
$$u(x,0) = g(x), \quad \left[\partial_{t_1} + \varepsilon \partial_{t_2}\right] u\Big|_{t_1 = t_2 = 0} = 0. \tag{3.4.2c}$$

As before, the solution of (3.4.2) is not unique and we will use this degree of freedom to eliminate the secular terms.

We try a regular asymptotic expansion of the form

$$u \sim u_0(x, t_1, t_2) + \varepsilon u_1(x, t_1, t_2) + \dots \quad \text{as } \varepsilon \longrightarrow 0. \tag{3.4.3}$$

The $\mathcal{O}(1)$ problem is

$$\partial_x^2 u_0 = \partial_{t_1}^2 u_0, \tag{3.4.4a}$$
$$u_0(x,0,0) = g(x), \quad \partial_{t_1} u_0(x,0,0) = 0. \tag{3.4.4b}$$

Separation of variables yields the general solution

$$u_0(x, t_1, t_2) = \sum_{n=1}^{\infty} \left[a_n(t_2)\sin(\lambda_n t_1) + b_n(t_2)\cos(\lambda_n t_1)\right]\sin(\lambda_n x), \quad \lambda_n = n\pi. \tag{3.4.5}$$

The initial conditions will be imposed once we determine $a_n(t_2)$ and $b_n(t_2)$. The $\mathcal{O}(\varepsilon)$ equation is

$$\begin{aligned}\partial_x^2 u_1 &= \partial_{t_1}^2 u_1 + 2\partial_{t_1}\partial_{t_2} u_0 + \partial_{t_1} u_0 \\ &= \partial_{t_1}^2 u_1 + \sum_{n=1}^{\infty} A_n(t_1, t_2)\sin(\lambda_n x),\end{aligned} \tag{3.4.6}$$

where

$$A_n = (2a_n' + a_n)\,\lambda_n \cos(\lambda_n t_1) - (2b_n' + b_n)\,\lambda_n \sin(\lambda_n t_1).$$

Given the zero boundary conditions in (3.4.1), it is appropriate to introduce the Fourier expansion

$$u_1 = \sum_{n=1}^{\infty} V_n(t_1, t_2)\sin(\lambda_n x).$$

Substituting this into (3.4.7) together with the expression of A_n, we obtain

$$\partial_{t_1}^2 V_n + \lambda_n^2 V_n = -\left(2a_n' + a_n\right)\lambda_n \cos(\lambda_n t_1) + \left(2b_n' + b_n\right)\lambda_n \sin(\lambda_n t_1).$$

The secular terms are eliminated provided

$$2a_n' + a_n = 0, \quad 2b_n' + b_n = 0,$$

and these have general solutions of the form

$$a_n(t_2) = a_n(0)e^{-t_2/2}, \quad b_n(t_2) = b_n(0)e^{-t_2/2}.$$

Finally, a first term approximation of the solution of (3.4.1) is

$$u(x,t) \sim \sum_{n=1}^{\infty} \left[a_n(0)e^{-\varepsilon t/2}\sin(\lambda_n t) + b_n(0)e^{-\varepsilon t/2}\cos(\lambda_n t)\right]\sin(\lambda_n x), \quad \lambda_n = n\pi. \tag{3.4.7}$$

Applying the initial condition in (3.4.4), we find that $a_n(0) = 0$ and

$$b_n(0) = 2\int_0^1 g(x)\sin(\lambda_n x)\,dx.$$

3.4.2 Nonlinear wave equation

Consider the nonlinear Klein-Gordon equation

$$\partial_x^2 u = \partial_t^2 u + u + \varepsilon u^3, \quad -\infty < x < \infty, \quad t > 0, \tag{3.4.8a}$$
$$u(x,0) = F(x), \quad \partial_t u(x,0) = G(x). \tag{3.4.8b}$$

It describes the motion of a string on an elastic foundation as well as the waves in a cold electron plasma.

As usual, let us consider (3.4.8) with $\varepsilon = 0$:

$$\partial_x^2 u = \partial_t^2 u + u, \quad -\infty < x < \infty, \quad t > 0, \tag{3.4.9a}$$
$$u(x,0) = F(x), \quad \partial_t u(x,0) = G(x). \tag{3.4.9b}$$

We guess a solution of the form $\exp\Big(i(kx - \omega t)\Big)$. This yields the dispersion relation

$$-k^2 = -\omega^2 + 1 \implies \omega = \pm\sqrt{1+k^2} = \pm\omega(k).$$

We may solve (3.4.9) using the spatial Fourier transform

$$\widehat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t)e^{-ikx}\,dx.$$

This produces an ODE for $\widehat{u}(k,t)$:

$$-k^2\widehat{u} = \partial_{tt}\widehat{u} + \widehat{u}, \tag{3.4.10a}$$
$$\widehat{u}(k,0) = \widehat{F}(k), \quad \partial_t\widehat{u}(k,0) = \widehat{G}(k). \tag{3.4.10b}$$

Solving (3.4.10) and applying the inverse Fourier transform we obtain the general solution of (3.4.9):

$$u(x,t) = \int_{-\infty}^{\infty} A(k)e^{i(kx-\omega(k)t)}\,dk + \int_{-\infty}^{\infty} B(k)e^{i(kx+\omega(k)t}\,dk, \tag{3.4.11}$$

where $A(k)$ and $B(k)$ are determined from the initial conditions in (3.4.10). This shows that the solution of (3.4.9) can be written as the superposition of the plane wave solutions $u_{\pm}(x,t) = \exp\Big(i(kx+\omega(k)t)\Big)$. We would like to investigate how the nonlinearity affects a right-moving plane wave $u(x,t) = \cos(kx-\omega t)$, where $k > 0$ and $\omega = \sqrt{1+k^2}$.

A regular asymptotic expansion of the form

$$u(x,t) \sim w_0(kx-\omega t) + \varepsilon w_1(x,t) + \dots$$

will lead to secular terms, and thus we use multiple scales to find an asymptotic approximation of the solution of (3.4.8). We take three independent variables

$$\theta = kx - \omega t, \quad x_2 = \varepsilon x, \quad t_2 = \varepsilon t.$$

The spatial and time derivatives become

$$\frac{\partial}{\partial x} \longrightarrow k\frac{\partial}{\partial \theta} + \varepsilon\frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial t} \longrightarrow -\omega\frac{\partial}{\partial \theta} + \varepsilon\frac{\partial}{\partial t_2}.$$

Consequently, the nonlinear Klein-Gordon equation becomes

$$\begin{aligned} \Big[k\partial_\theta + \varepsilon\partial_{x_2}\Big]^2 u &= \Big[-\partial_\theta + \varepsilon\partial_{t_2}\Big]^2 u + u + \varepsilon u^3 \\ \Big[k^2-\omega^2\Big]\partial_\theta^2 u + 2\varepsilon k\partial_{x_2}\partial_\theta u + 2\varepsilon\omega\partial_{t_2}\partial_\theta u &= u + \varepsilon u^3 + \mathcal{O}(\varepsilon^2)u \\ \Big[\partial_\theta^2 - 2\varepsilon\Big(k\partial_{x_2} + \omega\partial_{t_2}\Big)\partial_\theta + \mathcal{O}(\varepsilon^2)\Big]u + u + \varepsilon u^3 &= 0, \end{aligned} \tag{3.4.12}$$

where we use the dispersion relation $-k^2 = -\omega^2 + 1$. We assume a regular asymptotic expansion of the form

$$u(x,t) \sim u_0(\theta, x_2, t_2) + \varepsilon u_1(\theta, x_2, t_2) + \dots.$$

The $\mathcal{O}(1)$ equation is

$$\left(\partial_\theta^2 + 1\right) u_0 = 0,$$

and the general solution of this problem is

$$u_0 = A(x_2,t_2)\cos\Big(\theta + \phi(x_2,t_2)\Big).$$

The $\mathcal{O}(\varepsilon)$ equation is

$$\begin{aligned} \left(\partial_\theta^2 + 1\right) u_1 &= 2\left(k\partial_{x_2} + \omega\partial_{t_2}\right)\partial_\theta u_0 - u_0^3 \\ &= -2\Big[\left(k\partial_{x_2} + \omega\partial_{t_2}\right)A\Big]\sin(\theta+\phi) - \frac{1}{4}A^3\cos\Big(3(\theta+\phi)\Big) \\ &\quad - 2\Big[\left(k\partial_{x_2} + \omega\partial_{t_2}\right)\phi + \frac{3}{8}A^2\Big]A\cos(\theta+\phi). \end{aligned}$$

The secular terms are eliminated provided

$$\left(k\partial_{x_2} + \omega\partial_{t_2}\right)A = 0 \tag{3.4.13a}$$

$$\left(k\partial_{x_2} + \omega\partial_{t_2}\right)\phi + \frac{3}{8}A^2 = 0. \tag{3.4.13b}$$

These constitute the **amplitude-phase equations** and can be solved using characteristic coordinates. Specifically, let

$$r = \omega x_2 + kt_2 \quad \text{and} \quad s = \omega x_2 - kt_2.$$

With this (3.4.13) simplifies to

$$\begin{aligned}\partial_r A &= 0\\ \partial_r \phi &= -\frac{3}{16\omega k}A^2\end{aligned}$$

and solving this yields

$$A = A(s) \quad \text{and} \quad \phi = -\frac{3}{16\omega k}A^2 r + \phi_0(s).$$

Hence, a first term approximation of the solution of (3.4.8) is

$$u \sim A(wx_2 - kt_2)\cos\left[(kx - \omega t) - \frac{3}{16\omega k}(\omega x_2 + kt_2)A^2 + \phi_0(\omega x_2 - kt_2)\right]. \tag{3.4.14}$$

We can now attempt to answer our main question: how does the nonlinearity affects the plane wave solution? Consider the plane wave initial conditions

$$u(x,0) = \alpha\cos(kx) \quad \text{and} \quad \partial_t u(x,0) = \alpha\omega\sin(kx).$$

In multiple scale expansion, these translates to

$$u_0(\theta, x_2, 0) = \alpha\cos(\theta) \quad \text{and} \quad \partial_\theta u_0(\theta, x_2, 0) = -\alpha\sin(\theta).$$

Imposing these initial conditions on (3.4.14) we obtain

$$A(\omega x_2) = \alpha \quad \text{and} \quad \phi_0(\omega x_2) = \frac{3}{16k}A^2 x_2.$$

Thus, a first term approximation of the solution of (3.4.8) in this case is

$$\begin{aligned}u(x,t) &\sim \alpha\cos\left(kx - \left(1 + \frac{3\varepsilon\alpha^2}{16\omega^2}\right)\omega t\right)\\ &\sim \alpha\cos(kx - \widehat{\omega}t).\end{aligned}$$

We see that the nonlinearity increases the phase velocity since it increases from $c = \omega/k$ to $c = \widehat{\omega}/k$.

3.5 Pattern Formation and Amplitude Equations

3.5.1 Neural field equations on a ring

Consider a population of neurons distributed on the circle $S^1 = [0, \pi]$:

$$\frac{\partial a}{\partial t} = -a(\theta, t) + \frac{1}{\pi}\int_0^\pi w(\theta - \theta')f(a(\theta', t))\,d\theta' \tag{3.5.1a}$$

$$f(a) = \frac{1}{1+\exp(-\eta(a-k))}, \tag{3.5.1b}$$

where $a(\theta,t)$ denotes the activity at time t of a local population of cells at position $\theta \in [0,\pi)$, $w(\theta-\theta')$ is the strength of synaptic weights between cells at θ' and θ and the firing rate function f is a sigmoid function. Assuming w is an even π-periodic function, it can be expanded as a Fourier series:

$$w(\theta) = W_0 + 2\sum_{n\geq 1} W_n \cos(2\pi\theta), \quad W_n \in \mathbb{R}. \tag{3.5.2}$$

Suppose there exists a uniform equilibrium solution $\bar{a}$ of (3.5.1), satisfying

$$\bar{a} = f(\bar{a})\int_0^\pi \frac{w(\theta-\theta')}{\pi}\,d\theta' = f(\bar{a})W_0. \tag{3.5.3}$$

The stability of the equilibrium solution is determined by setting $a(\theta,t) = \bar{a} + a(\theta)e^{\lambda t}$ and linearising (3.5.1) about $\bar{a}$. Expanding f around $\bar{a}$ yields

$$f(\bar{a}+a(\theta)e^{\lambda t}) \approx f(\bar{a}) + f'(\bar{a})a(\theta)e^{\lambda t},$$

and we obtain the eigenvalue equation

$$\lambda a(\theta) = -a(\theta) + \frac{f'(\bar{a})}{\pi}\int_0^\pi w(\theta-\theta')a(\theta')\,d\theta' = La(\theta). \tag{3.5.4}$$

Since the linear operator L is compact on $L^2(S^1)$, it has a discrete spectrum with eigenvalues

$$\lambda_n = -1 + f'(\bar{a})W_n, \quad n \in \mathbb{Z},$$

and corresponding eigenfunctions

$$a_n(\theta) = z_n e^{2in\theta} + z_n^* e^{-2in\theta}.$$

These are obtained by integrating the eigenvalue equation against $\cos(2n\theta)$ over $[0,\pi]$: CHT: Check this again, unsure about this

$$\begin{aligned}
\lambda_n a_n &= -a_n + \frac{f'(\bar{a})}{\pi}\int_0^\pi \left(\int_0^\pi w(\theta-\theta')a(\theta')\,d\theta'\right)\cos(2n\theta)\,d\theta \\
&= -a_n + \frac{f'(\bar{a})}{\pi}\int_0^\pi a(\theta')\left(\int_0^\pi \sum_{m\in\mathbb{Z}} W_m \cos(2m(\theta-\theta'))\cos(2n\theta)\,d\theta\right)d\theta' \\
&= -a_n + \frac{f'(\bar{a})}{\pi}\sum_{m\in\mathbb{Z}}\int_0^\pi W_m a(\theta')\cos(2n\theta')\,d\theta'\int_0^\pi \cos(2m\theta)\cos(2n\theta) + \sin(2m\theta)\cos(2n\theta)\,d\theta \\
&= -a_n + \frac{f'(\bar{a})}{\pi}\sum_{m\in\mathbb{Z}} W_m a_m\left[\frac{\pi}{2}\delta_{\pm m,n}\right] \\
&= -a_n + \frac{f'(\bar{a})}{2}\left[W_n a_n + W_{-n}a_{-n}\right] \\
&= -a_n + f'(\bar{a})W_n a_n,
\end{aligned}$$

where

$$a_n = \int_0^\pi a(\theta)\cos(2n\theta)\,d\theta = a_{-n}.$$

The eigenvalue expression reveals the bifurcation parameter $\mu = f'(\bar{a})$. For sufficiently small μ, corresponding to a low activity state, $\lambda_n < 0$ for all n and the fixed point is stable. As μ increases beyond a critical value μ_c, the fixed point becomes unstable due to excitation of the eigenfunctions associated with the largest Fourier component of $w(\theta)$. Suppose that $W_1 = \max_m W_m$. Then $\lambda_n > 0$ for all n if and only if

$$1 < \mu W_n \leq \mu W_1 \implies \mu > \frac{1}{W_1} = \mu_c.$$

Consequently, for $\mu > \mu_c$, the excited modes will be

$$a(\theta) = ze^{2i\theta} + \bar{z}e^{-2i\theta} = 2|z|\cos(2(\theta - \theta_0)),$$

where $z = |z|e^{-2i\theta_0}$. We expect this mode to grow and stop at a maximum amplitude as μ approaches μ_c, mainly because of the saturation of f.

3.5.2 Derivation of amplitude equation using the Fredholm alternative

Unfortunately, the linear stability analysis breaks down for large amplitude of the activity profile. Suppose the system is just above the bifurcation point, *i.e.*

$$\mu - \mu_c = \varepsilon\Delta\mu, \quad 0 < \varepsilon \ll 1 \tag{3.5.5}$$

If $\Delta\mu = \mathcal{O}(1)$, then $\mu - \mu_c = \mathcal{O}(\varepsilon)$ and we can carry out a perturbation expansion in powers of ε. We first Taylor expand the nonlinear function f around $a = \bar{a}$:

$$f(a) - f(\bar{a}) = \mu(a - \bar{a}) + g_2(a - \bar{a})^2 + g_3(a - \bar{a})^3 + \mathcal{O}(a - \bar{a})^4. \tag{3.5.6}$$

Assume a perturbation expansion of the form

$$a = \bar{a} + \sqrt{\varepsilon}a_1 + \varepsilon a_2 + \varepsilon^{3/2}a_3 + \dots. \tag{3.5.7}$$

The dominant temporal behaviour just beyond bifurcation is the slow growth of the excited mode $e^{\varepsilon\Delta\mu t}$ and this motivates the introduction of a slow time scale $\tau = \varepsilon t$. Substituting (3.5.5), (3.5.6) and (3.5.7) into (3.5.1) yields

$$\begin{aligned}
\Big[\partial_t + \varepsilon\partial_\tau\Big]&\Big[\bar{a} + \sqrt{\varepsilon}a_1 + \varepsilon a_2 + \varepsilon^{3/2}a_3 + \dots\Big] \\
&= -\Big[\bar{a} + \sqrt{\varepsilon}a_1 + \varepsilon a_2 + \varepsilon^{3/2}a_3 + \dots\Big] \\
&\quad + \frac{1}{\pi}\int_0^\pi w(\theta - \theta')f(\bar{a})\,d\theta' \\
&\quad + \frac{1}{\pi}\int_0^\pi w(\theta - \theta')\Big(\mu_c + \varepsilon\Delta\mu\Big)\Big[\sqrt{\varepsilon}a_1 + \varepsilon a_2 + \varepsilon^{3/2}a_3 + \dots\Big]\,d\theta' \\
&\quad + \frac{1}{\pi}\int_0^\pi w(\theta - \theta')g_2\Big[\sqrt{\varepsilon}a_1 + \varepsilon a_2 + \dots\Big]^2\,d\theta' \\
&\quad + \frac{1}{\pi}\int_0^\pi w(\theta - \theta')g_3\Big[\sqrt{\varepsilon}a_1 + \varepsilon a_2 + \dots\Big]^3\,d\theta'
\end{aligned}$$

Define the linear operator $\widehat{L}$:

$$La(\theta) = -a(\theta) + \frac{\mu_c}{\pi}\int_0^\pi w(\theta-\theta')a(\theta')\,d\theta' = -a(\theta) + \mu_c\, w * a(\theta).$$

Collecting terms with equal powers of ε then leads to a hierarchy of equations of the form:

$$\begin{aligned}
\bar{a} &= W_0 f(\bar{a}) \\
\widehat{L}a_1 &= 0 \\
\widehat{L}a_2 &= V_2 := -g_2\, w * a_1^2 \\
\widehat{L}a_3 &= V_3 := \frac{\partial a_1}{\partial \tau} - \Delta\mu\, w * a_1 - 2g_2\, w * (a_1 a_2) - g_3\, w * a_1^3.
\end{aligned}$$

The $\mathcal{O}(1)$ equation determines the fixed point $\bar{a}$. The $\mathcal{O}(\sqrt{\varepsilon})$ equation has solutions of the form

$$a_1 = z(\tau)e^{2i\theta} + z^*(\tau)e^{-2i\theta}.$$

A dynamical equation for $z(\tau)$ can be obtained by deriving solvability conditions for the higher-order equations using Fredholm alternative. These equations have the general form

$$\widehat{L}a_n = V_n(\bar{a}, a_1, \ldots, a_{n-1}), \quad n \geq 2.$$

For any two periodic functions U, V, define the inner product

$$\langle U, V\rangle = \frac{1}{\pi}\int_0^\pi U^*(\theta)V(\theta)\,d\theta.$$

Using integration by parts, it is easy to see that $\widehat{L}$ is self-adjoint with respect to this particular inner product and since $\widehat{L}\tilde{a} = 0$ for $\tilde{a} = e^{\pm 2i\theta}$, we have

$$\langle \tilde{a}, \widehat{L}a_n\rangle = \langle \widehat{L}\tilde{a}, a_n\rangle = 0.$$

Since $\widehat{L}a_n = V_n$, it follows from the Fredholm alternative that the set of solvability conditions are

$$\langle \tilde{a}, V_n\rangle = 0 \quad \text{for } n \geq 2.$$

The $\mathcal{O}(\varepsilon)$ solvability condition $\langle \tilde{a}, V_2\rangle = 0$ is automatically satisfied. The $\mathcal{O}(\varepsilon^{3/2})$ solvability condition can be expanded into

$$\langle \tilde{a}, \partial_\tau a_1 - \Delta\mu\, w * a_1\rangle = g_3\langle \tilde{a}, w * a_1^3\rangle + 2g_2\langle \tilde{a}, w * (a_1 a_2)\rangle. \tag{3.5.8}$$

Taking $\tilde{a} = e^{2i\theta}$ then generates a cubic amplitude for z. First, we have

$$\langle e^{2i\theta}, \partial_\tau a_1\rangle = \frac{1}{\pi}\int_0^\pi e^{-2i\theta}\left(\frac{dz}{d\tau}e^{2i\theta} + \frac{dz^*}{d\tau}e^{-2i\theta}\right)d\theta = \frac{dz}{d\tau} \tag{3.5.9}$$

To deal with the convolution terms, observe that since w is even, for any function $b(\theta)$ we have

$$\begin{aligned}
\langle e^{2i\theta}, w * b\rangle &= \langle w * e^{2i\theta}, b\rangle \\
&= \frac{1}{\pi}\int_0^\pi \left(\frac{1}{\pi}\int_0^\pi w(\theta-\theta')e^{-2i\theta'}\,d\theta'\right) b(\theta)\,d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\int_0^\pi \left(\frac{1}{\pi}\int_0^\pi \sum_{n\geq 1} 2W_n \cos(2n(\theta-\theta'))e^{-2i\theta'}\,d\theta'\right) b(\theta)\,d\theta \\
&= \frac{1}{\pi}\int_0^\pi \left(\frac{1}{\pi}\int_0^\pi \sum_{n\geq 1} W_n \left(e^{2in(\theta-\theta')} + e^{-2in(\theta-\theta')}\right)e^{-2i\theta'}\,d\theta'\right) b(\theta)\,d\theta \\
&= \frac{1}{\pi}\int_0^\pi \left(\frac{1}{\pi} W_1 e^{-2i\theta}\int_0^\pi d\theta'\right) b(\theta)\,d\theta \\
&= \frac{1}{\pi}\int_0^\pi W_1 e^{-2i\theta} b(\theta)\,d\theta \\
&= W_1\langle e^{2i\theta}, b\rangle.
\end{aligned}
$$

Set $W_1 = \mu_c^{-1}$. From the identity above we then have

$$
\begin{aligned}
\langle e^{2i\theta}, \Delta\mu\, w * a_1\rangle &= \Delta\mu W_1 \langle e^{2i\theta}, a_1\rangle \\
&= \frac{\Delta\mu}{\mu_c \pi}\int_0^\pi \int_0^\pi e^{-2i\theta}\left(ze^{2i\theta} + z^* e^{-2i\theta}\right)d\theta \\
&= \mu_c^{-1}\Delta\mu z
\end{aligned} \tag{3.5.10}
$$

and

$$
\begin{aligned}
\langle e^{2i\theta}, w * a_1^3\rangle &= W_1\langle e^{2i\theta}, a_1^3\rangle \\
&= \frac{1}{\mu_c\pi}\int_0^\pi e^{-2i\theta}\left(ze^{2i\theta} + z^* e^{-2i\theta}\right)^3 d\theta \\
&= \frac{1}{\mu_c\pi}\int_0^\pi e^{-2i\theta}\left(z^3 e^{6i\theta} + 3z^2 z^* e^{2i\theta} + 3zz^* e^{-2i\theta} + (z^*)^3 e^{-6i\theta}\right) d\theta \\
&= 3\mu_c^{-1} z^2 z^* = 3\mu_c^{-1} z|z|^2.
\end{aligned} \tag{3.5.11}
$$

The next step is to determine a_2. From the $\mathcal{O}(\varepsilon)$ equation we have

$$
\begin{aligned}
-\widehat{L}a_2 &= a_2 - \frac{\mu_c}{\pi}\int_0^\pi w(\theta-\theta')a_2(\theta')\,d\theta' \\
&= \frac{g_2}{\pi}\int_0^\pi w(\theta-\theta')a_1^2(\theta')\,d\theta' \\
&= \frac{g_2}{\pi}\int_0^\pi \left\{W_0 + \sum_{n\geq 1} W_n\left(e^{2in(\theta-\theta')} + e^{-2in(\theta-\theta')}\right)\right\}\left[z^2 e^{4i\theta'} + 2|z|^2 + (z^*)^2 e^{-4i\theta'}\right] d\theta' \\
&= g_2\left[2|z|^2 W_0 + z^2 W_2 e^{4i\theta} + (z^*)^2 W_2 e^{-4i\theta}\right].
\end{aligned} \tag{3.5.12}
$$

Let

$$
a_2(\theta) = A_+ e^{4i\theta} + A_- e^{-4i\theta} + A_0 + \zeta a_1(\theta). \tag{3.5.13}
$$

The constant ζ remains undetermined at this order of perturbation but does not appear in the amplitude equation for $z(\tau)$. Substituting (3.5.13) into (3.5.12) yields

$$
A_+ = \frac{g_2 z^2 W_2}{1-\mu_c W_2}, \quad A_- = \frac{g_2 (z^*)^2 W_2}{1-\mu_c W_2}, \quad A_0 = \frac{2g_2|z|^2 W_0}{1-\mu_c W_0}. \tag{3.5.14}
$$

Consequently,

$$\begin{aligned}
\langle e^{2i\theta}, w * (a_1 a_2)\rangle &= W_1 \langle e^{2i\theta}, a_1 a_2\rangle \\
&= \frac{1}{\mu_c \pi}\int_0^{\pi} e^{-2i\theta}\Big(z e^{2i\theta} + z^* e^{-2i\theta}\Big)\Big(A_+ e^{4i\theta} + A_- e^{-4i\theta} + A_0 + \zeta a_1(\theta)\Big)\, d\theta \\
&= \mu_c^{-1}\Big[z^* A_+ + z A_0\Big] \\
&= \mu_c^{-1}\left[\frac{g_2 z^* |z|^2 W_2}{1-\mu_c W_2} + \frac{g_2 z |z|^2 W_0}{1-\mu_c W_0}\right] && (3.5.15) \\
&= z|z|^2 g_2 \mu_c^{-1}\left[\frac{W_2}{1-\mu_c W_2} + \frac{2W_0}{1-\mu_c W_0}\right]. && (3.5.16)
\end{aligned}$$

Finally, substituting (3.5.9), (3.5.10), (3.5.11) and (3.5.16) into the $\mathcal{O}(\varepsilon^{3/2})$ solvability condition (3.5.8), we obtain the Stuart-Landau equation

$$\frac{dz}{d\tau} = z(\Delta\mu - \Lambda |z|^2), \tag{3.5.17}$$

where

$$\Lambda = -3g_3 - 2g_2^2\left[\frac{W_2}{1-\mu_c W_2} + \frac{2W_0}{1-\mu_c W_0}\right]. \tag{3.5.18}$$

Note that we also absorbed a factor of μ_c into τ.

3.6 Problems

1. Find a first-term expansion of the solution of the following problems using two time scales.

   (a) $y'' + \varepsilon(y')^3 + y = 0$, $y(0) = 0$, $y'(0) = 1$.

Solution: We introduce a slow scale $\tau = \varepsilon t$ and an asymptotic expansion

$$y \sim y_0(t,\tau) + \varepsilon y_1(t,\tau) + \dots.$$

The original problem becomes

$$\Big[\partial_t^2 + 2\varepsilon\partial_t\partial_\tau + \varepsilon^2\partial_\tau^2\Big]\Big(y_0 + \varepsilon y_1 + \dots\Big) + \varepsilon\Big[\Big[\partial_t + \varepsilon\partial_\tau\Big]\Big(y_0 + \varepsilon y_1 + \dots\Big)\Big]^3 + \Big(y_0 + \varepsilon y_1 + \dots\Big) = 0,$$

with boundary conditions

$$\begin{aligned}
\Big(y_0 + \varepsilon y_1 + \dots\Big)(0,0) &= 0 \\
\Big[\partial_t + \varepsilon\partial_\tau\Big]\Big(y_0 + \varepsilon y_1 + \dots\Big)(0,0) &= 1.
\end{aligned}$$

The $\mathcal{O}(1)$ problem is

$$\left(\partial_t^2 + 1\right) y_0 = 0, \quad y_0(0,0) = \partial_t y_0(0,0) = 0, \tag{3.6.1}$$

and its general solution is

$$y_0(t,\tau) = A(\tau)e^{it} + A^*(\tau)e^{-it}, \tag{3.6.2}$$

where $A(\tau)$ is complex function of τ. The $\mathcal{O}(\varepsilon)$ equation is

$$\begin{aligned}
\left(\partial_t^2 + 1\right) y_1 &= -2\partial_t\partial_\tau y_0 - \left(\partial_t y_0\right)^3 \\
&= -2i\left[A_\tau e^{it} - A_\tau^* e^{-it}\right] - (i^3)\left[Ae^{it} - A^* e^{-it}\right]^3 \\
&= -2i\left[A_\tau e^{it} - A_\tau^* e^{-it}\right] + i\left[A^3 e^{3it} - 3A^2 A^* e^{it} + 3A(A^*)^2 e^{-it} + (A^*)^3 e^{-3it}\right] \\
&= -i\left[2A_\tau + 3A|A|^2\right]e^{it} + i\left[2A_\tau^* + 3A^*|A|^2\right]e^{-it} + i\left[A^3 e^{3it} + (A^*)^3 e^{-3it}\right] \\
&= F(\tau)e^{it} + F^*(\tau)e^{-it} + i\left[A^3 e^{3it} + (A^*)^3 e^{-3it}\right].
\end{aligned}$$

The secular terms are eliminated provided $F(\tau) = 0$. Writing $A(\tau) = R(\tau)e^{i\theta(\tau)}$, $F(\tau)$ becomes

$$2\left(R_\tau e^{i\theta} + iR\theta_\tau e^{i\theta}\right) + 3Re^{i\theta}R^2 = 0,$$

or

$$2\left(R_\tau + iR\theta_\tau\right) + 3R^3 = 0.$$

Consequently, we have

$$\theta_\tau = 0 \implies \theta(\tau) = \theta_0$$

and

$$2R_\tau + 3R^3 = 0 \implies \frac{2R_\tau}{R^3} = -3 \implies \frac{1}{R^2} = 3\tau + C \implies R(\tau) = \frac{1}{\sqrt{3\tau + C}}.$$

Therefore, (3.6.2) becomes

$$\begin{aligned}
y_0(t,\tau) &= R(\tau)e^{i(t+\theta_0)} + R(\tau)e^{-i(t+\theta_0)} \\
&= 2R(\tau)\cos(t + \theta_0).
\end{aligned}$$

We now impose the initial conditions from (3.6.1):

$$\begin{aligned}
y_0(0,0) = 0 &\implies 2R(0)\cos(\theta_0) = 0 \\
\partial_t y_0(0,0) = 1 &\implies -2R(0)\sin(\theta_0) = 1,
\end{aligned}$$

which means

$$R(0)e^{i\theta_0} = -\frac{i}{2} = \frac{1}{\sqrt{C}}e^{i\theta_0} \implies C = 4 \quad \text{and } \theta_0 = \frac{3\pi}{2}.$$

Hence, a first-term approximation of the solution of the original problem is

$$y \sim \frac{2\cos(t + 3\pi/2)}{\sqrt{3\varepsilon t + 4}} \sim \frac{2\sin(t)}{\sqrt{3\varepsilon t + 4}}.$$

(b) $\varepsilon y'' + \varepsilon\kappa y' + y + \varepsilon y^3 = 0$, $y(0) = 0$, $y'(0) = 1$, $\kappa > 0$.

Solution: The equation appears to have a boundary layer, but it does not in this case since ε appears on y' as well. Let $T = t/\sqrt{\varepsilon}$ and $Y(T) = y(t) = y(\sqrt{\varepsilon}T)$, then

$$\frac{d}{dt} = \frac{1}{\sqrt{\varepsilon}}\frac{d}{dT}$$

and the original problem becomes

$$\partial_T^2 Y + \sqrt{\varepsilon}\kappa\partial_T Y + Y + \varepsilon Y^3 = 0 \tag{3.6.3a}$$
$$Y(0) = 0, \quad \partial_T Y(0) = \sqrt{\varepsilon}. \tag{3.6.3b}$$

Since one of the boundary conditions is of $\mathcal{O}(\sqrt{\varepsilon})$, we take the slow scale to be $\tau = \sqrt{\varepsilon}T = t$ and the fast scale to be $T = t/\sqrt{\varepsilon}$. Assuming an asymptotic expansion of the form

$$Y \sim Y_0(T,\tau) + \sqrt{\varepsilon}Y_1(T,\tau) + \dots. \tag{3.6.4}$$

Substituting (3.6.4) into (3.6.3) we obtain

$$\left[\partial_T^2 + 2\sqrt{\varepsilon}\partial_T\partial_\tau + \varepsilon\partial_\tau^2\right]\left(Y_0 + \sqrt{\varepsilon}Y_1 + \dots\right) + \sqrt{\varepsilon}\kappa\left[\partial_T + \sqrt{\varepsilon}\partial_\tau\right]\left(Y_0 + \sqrt{\varepsilon}Y_1 + \dots\right)$$
$$+ \left(Y_0 + \sqrt{\varepsilon}Y_1 + \dots\right) + \varepsilon\left(Y_0 + \sqrt{\varepsilon}Y_1 + \dots\right)^3 = 0,$$

with boundary conditions

$$\left(Y_0 + \sqrt{\varepsilon}Y_1 + \dots\right)(0,0) = 0$$
$$\left[\partial_T + \sqrt{\varepsilon}\partial_\tau\right]\left(Y_0 + \sqrt{\varepsilon}Y_1 + \dots\right)(0,0) = \sqrt{\varepsilon}.$$

The $\mathcal{O}(1)$ problem is

$$\left(\partial_T^2 + 1\right)Y_0 = 0, \quad Y_0(0,0) = \partial_T Y_0(0,0) = 0,$$

and its general solution is

$$Y_0(T,\tau) = A(\tau)e^{iT} + A^*(\tau)e^{-iT}.$$

The $\mathcal{O}(\sqrt{\varepsilon})$ equation is

$$\begin{aligned}\left(\partial_T^2 + 1\right)Y_1 &= -2\partial_T\partial_\tau Y_0 - \kappa\partial_T Y_0 \\ &= -2i\left[A_\tau e^{it} - A_\tau^* e^{-it}\right] - \kappa i\left[Ae^{it} - A^* e^{-it}\right] \\ &= -i\left[2A_\tau + \kappa A\right]e^{it} + i\left[2A_\tau^* + \kappa A^*\right]e^{-it} \\ &= F(\tau)e^{it} + F^*(\tau)e^{-it}.\end{aligned}$$

The secular terms are eliminated provided $F(\tau) = 0$, *i.e.*

$$2A_\tau + \kappa A = 0 \implies A(\tau) = A(0)e^{-\kappa\tau/2}.$$

It can be easily seen from the initial conditions of the $\mathcal{O}(1)$ problem that $A(0) = 0$, and so $Y_0 \equiv 0$. Before we proceed any further, note that

$$Y_1(T, \tau) = B(\tau)e^{iT} + B^*(\tau)e^{-iT}.$$

The $\mathcal{O}(\varepsilon)$ equation is

$$\begin{aligned}\left(\partial_T^2 + 1\right)Y_2 &= -2\partial_T\partial_\tau Y_1 - \partial_\tau^2 Y_0 - \kappa\left(\partial_T Y_1 + \partial_\tau Y_0\right) - Y_0^3 \\ &= -2\partial_T\partial_\tau Y_1 - \kappa\partial_T Y_1.\end{aligned}$$

This has the same structure as the $\mathcal{O}(\sqrt{\varepsilon})$ equation and it should be clear then that the secular terms are eliminated provided

$$2B_\tau + \kappa B = 0 \implies B(\tau) = B(0)e^{-\kappa\tau/2}.$$

Imposing the initial condition $Y_1(0, 0) = 0$ and $(\partial_T Y_1 + \partial_\tau Y_0)(0, 0) = \partial_T Y_1(0, 0) = 1$, we obtain

$$\begin{aligned}B(0) + B^*(0) &= 0 \\ i\left[B(0) - B^*(0)\right] &= 1,\end{aligned}$$

which gives $B(0) = -i/2$. Hence, the $\mathcal{O}(\sqrt{\varepsilon})$ solution is

$$\begin{aligned}Y_1(T, \tau) &= B(0)e^{-\kappa\tau/2}e^{iT} + B^*(0)e^{-\kappa\tau/2}e^{-iT} \\ &= e^{-\kappa\tau/2}\left[-\frac{i}{2}e^{iT} + \frac{i}{2}e^{-iT}\right] \\ &= e^{-\kappa\tau/2}\sin(T)\end{aligned}$$

and a first-term approximation of the solution of the original problem is

$$y(t) = Y(T) \sim e^{-\kappa t/2}\sin\left(\frac{t}{\sqrt{\varepsilon}}\right).$$

2. In the study of Josephson junctions, the following problem appears

$$\phi'' + \varepsilon\left(1 + \gamma\cos\phi\right)\phi' + \sin\phi = \varepsilon\alpha, \quad \phi(0) = 0, \quad \phi'(0) = 0, \quad \gamma > 0. \tag{3.6.5}$$

Use the method of multiple scales to find a first-term approximation of $\phi(t)$.

Solution: With the slow scale $\tau = \varepsilon t$, (3.6.5) becomes

$$\begin{cases} \left[\partial_t^2 + 2\varepsilon\partial_t\partial_\tau + \varepsilon^2\partial_\tau^2\right]\phi + \varepsilon\left(1 + \gamma\cos\phi\right)\left[\partial_t + \varepsilon\partial_\tau\right]\phi + \sin\phi = \varepsilon\alpha, \\ \phi(0,0) = 0, \quad \left[\partial_t + \varepsilon\partial_\tau\right]\phi(0,0) = 0, \quad \gamma > 0. \end{cases} \tag{3.6.6}$$

Assume an asymptotic expansion of the form

$$\phi \sim \phi_0(t,\tau) + \varepsilon\phi_1(t,\tau) + \varepsilon^2\phi_2(t,\tau) + \dots. \tag{3.6.7}$$

Substituting (3.6.7) into (3.6.6) and expanding both $\sin(\phi)$ and $\cos(\phi)$ around $\phi = \phi_0$ we obtain:

$$\begin{aligned} &\left[\partial_t^2 + 2\varepsilon\partial_t\partial_\tau + \varepsilon^2\partial_\tau^2\right]\left(\phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \dots\right) \\ &+ \varepsilon\Big(1 + \gamma\left[\cos\phi_0 - \sin\phi_0\left(\varepsilon\phi_1 + \varepsilon^2\phi_2 + \dots\right)\right]\Big)\left[\partial_t + \varepsilon\partial_\tau\right]\left(\phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \dots\right) \\ &+ \left[\sin\phi_0 + \cos\phi_0\left(\varepsilon\phi_1 + \varepsilon^2\phi_2 + \dots\right)\right] = \varepsilon\alpha, \end{aligned}$$

with boundary conditions

$$\begin{aligned} \left(\phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \dots\right)(0,0) &= 0 \\ \left[\partial_t + \varepsilon\partial_\tau\right]\left(\phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \dots\right)(0,0) &= 0. \end{aligned}$$

The $\mathcal{O}(1)$ problem is

$$\partial_t^2\phi_0 + \sin\phi_0 = 0, \quad \phi_0(0,0) = 0, \quad \partial_t\phi_0(0,0) = 0.$$

To solve this nonlinear problem ,we approximate $\sin\phi_0 \approx \phi_0$ and the general solution of the problem is approximately

$$\phi_0(t,\tau) \approx A(\tau)\cos(t) + B(\tau)\sin(t). \tag{3.6.8}$$

The initial conditions gives $A(0) = 0 = B(0)$.

The $\mathcal{O}(\varepsilon)$ equation is

$$\partial_t^2\phi_1 + 2\partial_t\partial_\tau\phi_0 + \left(1 + \gamma\cos\phi_0\right)\partial_t\phi_0 + \left(\cos\phi_0\right)\phi_1 = \alpha,$$

with boundary conditions

$$\phi_1(0,0) = 0, \quad \partial_t\phi_1(0,0) = -\partial_\tau\phi_0(0,0).$$

We approximate $\cos\phi_0 \approx 1$ and substitute the expression (3.6.8) for ϕ_0:

$$\begin{aligned}\partial_t^2\phi_1 + \phi_1 &= -2\partial_t\partial_\tau\phi_0 - (1+\gamma)\,\partial_t\phi_0 + \alpha \\ &= -2\left(-A'\sin(t) + B'\cos(t)\right) - (1+\gamma)\left(-A\sin t + B\cos(t)\right) + \alpha \\ &= \left[2A' + A(1+\gamma)\right]\sin(t) - \left[2B' + B(1+\gamma)\right]\cos(t) + \alpha.\end{aligned}$$

The secular terms are eliminated provided the coefficients of $\cos(t)$ and $\sin(t)$ vanish. This yields two initial value problems

$$\begin{cases} 2A' + A(1+\gamma) = 0, & A(0) = 0 \\ 2B' + B(1+\gamma) = 0, & B(0) = 0 \end{cases}$$

which has solutions $A(\tau) = B(\tau) \equiv 0$. It follows from (3.6.8) that $\phi_0 \equiv 0$ and we need to investigate the $\mathcal{O}(\varepsilon^2)$ problem. The general solution of the $\mathcal{O}(\varepsilon)$ problem is

$$\phi_1(t,\tau) \approx C(\tau)\cos(t) + D(\tau)\sin(t) + \alpha, \tag{3.6.9}$$

and the initial conditions gives $C(0) = -\alpha$ and $D(0) = 0$.

The $\mathcal{O}(\varepsilon^2)$ equation is

$$\begin{aligned}\partial_t^2\phi_2 + 2\partial_t\partial_\tau\phi_1 + \partial_\tau^2\phi_0 + (1+\gamma\cos\phi_0)\left(\partial_t\phi_1 + \partial_\tau\phi_0\right) \\ - \gamma\left(\sin\phi_0\right)\phi_1\partial_t\phi_0 + \left(\cos\phi_0\right)\phi_2 = 0.\end{aligned}$$

Simplifying using $\phi_0 \equiv 0$ we obtain

$$\begin{aligned}\partial_t^2\phi_2 + \phi_2 &= -2\partial_t\partial_\tau\phi_1 - (1+\gamma)\partial_t\phi_1 \\ &= -2\left(-C'\sin(t) + B'\cos(t)\right) - (1+\gamma)\left(-A\sin(t) + B\cos(t)\right) \\ &= \left[2C' + C(1+\gamma)\right]\sin(t) - \left[2D' + D(1+\gamma)\right]\cos(t).\end{aligned}$$

The secular terms are eliminated provided the coefficients of $\cos(t)$ and $\sin(t)$ vanish. This yields two initial value problems

$$\begin{cases} 2C' + C(1+\gamma) = 0, & C(0) = -\alpha \\ 2D' + D(1+\gamma) = 0, & D(0) = 0 \end{cases}$$

and the general solutions are $D(\tau) \equiv 0$ and

$$C(\tau) = -\alpha\exp\left(-\left(\frac{1+\gamma}{2}\right)\tau\right).$$

Hence, a first-term approximation of the solution of the original problem (3.6.5) is

$$\phi \sim \varepsilon\left(\alpha - \alpha e^{-(1+\gamma)\varepsilon t/2}\cos(t)\right) \sim \varepsilon\alpha\left(1 - e^{-(1+\gamma)\varepsilon t/2}\cos(t)\right).$$

3. Consider the equation

$$\ddot{x} + \dot{x} = -\varepsilon(x^2 - x), \quad 0 < \varepsilon \ll 1. \tag{3.6.10}$$

Use the method of multiple scales to show that

$$x_0(t,\tau) = A(\tau) + B(\tau)e^{-t},$$

with $\tau = \varepsilon t$, and identify any resonant terms at $\mathcal{O}(\varepsilon)$. Show that the non-resonance condition is $\partial_\tau A = A - A^2$ and describe the asymptotic behaviour of solutions.

Solution: With the slow scale $\tau = \varepsilon t$ and assuming an asymptotic expansion of the form

$$x(t,\tau) \sim x_0(t,\tau) + \varepsilon x_1(t,\tau) + \dots,$$

the differential equation (3.6.10) becomes

$$\begin{aligned}\Big[\partial_t^2 + 2\varepsilon\partial_t\partial_\tau + \varepsilon^2\partial_\tau^2\Big]\Big(x_0 + \varepsilon x_1 + \dots\Big) + \Big[\partial_t + \varepsilon\partial_\tau\Big]\Big(x_0 + \varepsilon x_1 + \dots\Big) \\ = -\varepsilon\Big[\Big(x_0 + \varepsilon x_1 + \dots\Big)^2 - \Big(x_0 + \varepsilon x_1 + \dots\Big)\Big] \\ = -\varepsilon\Big[x_0^2 - x_0\Big] + \mathcal{O}(\varepsilon^2).\end{aligned}$$

The $\mathcal{O}(1)$ equation is

$$\partial_t^2 x_0 + \partial_t x_0 = 0,$$

and its general solution is

$$x_0(t,\tau) = A(\tau) + B(\tau)e^{-t}.$$

The $\mathcal{O}(\varepsilon)$ equation is

$$\begin{aligned}\partial_t^2 x_1 + \partial_t x_1 &= -2\partial_t\partial_\tau x_0 - \partial_\tau x_0 - (x_0^2 - x_0) \\ &= -2\Big[-B_\tau e^{-t}\Big] - \Big[A_\tau + B_\tau e^{-t}\Big] - \Big[(A + Be^{-t})^2 - (A + Be^{-t})\Big] \\ &= -\Big[A^2 - A + A_\tau\Big] - e^{-t}\Big[B_\tau - 2B_\tau + 2AB - B\Big] - B^2 e^{-2t} \\ &= F(\tau) + G(\tau)e^{-t} + H(\tau)e^{-2t}.\end{aligned}$$

Since the first two terms belongs to the kernel of the homogeneous operator, the corresponding particular solution has the form $F(\tau)$ and $G(\tau)te^{-t}$ and only the first one blows up as $t \longrightarrow \infty$, since

$$G(\tau)te^{-t} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty.$$

Hence, the non-resonance condition is $F(\tau) = 0$, or

$$\partial_\tau A = A - A^2. \tag{3.6.11}$$

A phase-plane analysis shows that the system (3.6.11) has an unstable fixed point at $A = 0$ and a stable fixed point at $A = 1$. Thus, we conclude that $A(\tau) \longrightarrow 1$ as $\tau \longrightarrow \infty$, provided $A(0) > 0$.

4. Consider the differential equation

$$\ddot{x} + x = -\varepsilon f(x,\dot{x}), \quad \text{with } |\varepsilon| \ll 1.$$

Let $y = \dot{x}$.

(a) Show that if $E(t) = E(x(t), y(t)) = (x(t)^2 + y(t)^2)/2$, then

$$\dot{E} = -\varepsilon f(x, y)y.$$

Hence, show that $E(t)$ is approximately 2π-periodic with $x = A_0\cos(t) + \mathcal{O}(\varepsilon)$ provided

$$\int_0^{2\pi} f(A_0\cos\tau, -A_0\sin\tau)\sin\tau\, d\tau = 0.$$

Solution: With $y = \dot{x}$, we have

$$\dot{y} = \ddot{x} = -x - \varepsilon f(x, \dot{x}) = -x - \varepsilon f(x, y).$$

Therefore

$$\begin{aligned}\dot{E}(x, y) &= x\dot{x} + y\dot{y} \\ &= xy + y\left(-x - \varepsilon f(x, y)\right) \\ &= -\varepsilon f(x, y)y.\end{aligned}$$

This means that to $\mathcal{O}(1)$, $\dot{E} = 0$ which implies an unperturbed solution of the form $x_0(t) = A_0\cos(t + \theta_0) = A_0\cos(t)$. WLOG we may take θ_0, as we can shift time to eliminate any phase shift because we are dealing with an autonomous system. Assume asymptotic expansions for both $E(x, y)$ and $x(t)$:

$$\begin{aligned}x &\sim x_0 + \varepsilon x_1 + \dots \\ E &\sim E_0 + \varepsilon E_1(t) + \dots.\end{aligned}$$

From the expression of $\dot{E}(x, y)$, the $\mathcal{O}(\varepsilon)$ equation is

$$\begin{aligned}\frac{dE_1}{dt} &= -f(x, y)y = -f(x_0 + \varepsilon x_1 + \dots, \dot{x_0} + \varepsilon\dot{x_1} + \dots)\Big(\dot{x_0} + \varepsilon\dot{x_1} + \dots\Big) \\ &= -f(x_0, \dot{x_0})\dot{x_0} + \mathcal{O}(\varepsilon).\end{aligned}$$

Therefore, to $\mathcal{O}(1)$,

$$\begin{aligned}E_1(t) &= E_1(0) - \int_0^t f(x_0(\tau), \dot{x_0}(\tau))\dot{x_0}(\tau)\, d\tau \\ &= E_1(0) + A_0\int_0^t f(A_0\cos(\tau), -A_0\sin(\tau))\sin(\tau)\, d\tau.\end{aligned}$$

If t is a multiple of 2π, say $t = 2\pi n$, then

$$E_1(2\pi n) = E_1(0) + nA_0\int_0^{2\pi} f(A_0\cos(\tau), -A_0\sin(\tau))\, d\tau$$

and we deduce that E_1 is approximately 2π-periodic if and only if

$$\int_0^{2\pi} f(A_0\cos(\tau), -A_0\sin(\tau))\sin(\tau)\,d\tau = 0.$$

(b) Suppose that the periodicity condition on part (a) does not hold. Let $E_n = E(x(2\pi n), y(2\pi n))$. Show that to lowest order E_n satisfies a difference equation of the form

$$E_{n+1} = E_n + \varepsilon F(E_n),$$

with

$$F(E_n) = \int_0^{2\pi} \sqrt{2E_n} f\left(\sqrt{2E_n}\cos\tau, -\sqrt{2E_n}\sin\tau\right)\sin\tau\,d\tau.$$

Hint: Take $x \sim A\cos t$ with $A = \sqrt{2E}$ slowly varying over a single period of length 2π.

Solution: Since

$$E(t) \sim E_0 + \varepsilon E_1(t) \sim \frac{A_0^2}{2},$$

we have

$$A_0(t) \approx \sqrt{2E(t)} + \mathcal{O}(\varepsilon).$$

From part (a), we then have

$$E(t+2\pi) \sim E_0 + \varepsilon E_1(t+2\pi)$$

(c) Hence, deduce that a periodic orbit with approximate amplitude $A^* = \sqrt{2E^*}$ exists if $F(E^*) = 0$ and this orbit is stable if

$$\varepsilon\frac{dF}{dE}(E^*) < 0.$$

Hint: Spiralling orbits close to the periodic orbit $x = A^*\cos(t) + \mathcal{O}(\varepsilon)$ can be approximated by a solution of the form $x = A\cos(t) + \mathcal{O}(\varepsilon)$.

Solution: From part (b), we have a one-dimensional map

(d) Using the above result, find the approximate amplitude of the periodic orbit of the Van der Pol equation

$$\ddot{x} + x + \varepsilon(x^2 - 1)\dot{x} = 0$$

and verify that it is stable.

Solution: In this case we have $f(x, y) = (x^2 - 1)y$ and so

$$\begin{aligned} F(E_n) &= \int_0^{2\pi} \sqrt{2E_n}\Big[2E_n\cos^2(\tau) - 1\Big]\Big[-\sqrt{2E_n}\sin(\tau)\Big]\sin(\tau)\,d\tau \\ &= \int_0^{2\pi} \Big[1 - 2E_n\cos^2(\tau)\Big]2E_n\sin^2(\tau)\,d\tau \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi} 2E_n \sin^2(\tau) - 4E_n^2 \sin^2(\tau)\cos^2(\tau)\, d\tau \\
&= \int_0^{2\pi} E_n\Big[1-\cos(2\tau)\Big] - E_n^2 \sin^2(2t)\, d\tau \\
&= \int_0^{2\pi} E_n\Big[1-\cos(2\tau)\Big] - \frac{E_n^2}{2}\Big[1-\cos(4\tau)\Big]\, d\tau \\
&= 2\pi\Big[E_n - \frac{E_n^2}{2}\Big] \\
&= \pi E_n\Big[2 - E_n\Big].
\end{aligned}
$$

Thus the zeros of F are $E^* = 0, 2$ and the approximate amplitude of the periodic orbit of the Van der Pol equation is $A^* = \sqrt{2E^*} = 2$. This orbit is stable since

$$
F'(E_n) = 2\pi(1-E_n) \implies F'(2) = -2\pi < 0.
$$

5. Consider the Van der Pol equation

$$
\ddot{x} + x + \varepsilon(x^2-1)\dot{x} = \Gamma\cos(\omega t), \quad 0 < \varepsilon \ll 1,
$$

with $\Gamma = \mathcal{O}(1)$ and $\omega \neq 1/3, 1, 3$. Use the method of multiple scales to show that the solution is attracted to

$$
x(t) = \left(\frac{\Gamma}{1-\omega^2}\right)\cos(\omega t) + \mathcal{O}(\varepsilon)
$$

when $\Gamma^2 \geq 2(1-\omega^2)^2$ and

$$
x(t) = 2\left[1 - \frac{\Gamma^2}{2(1-\omega^2)^2}\right]^{1/2}\cos t + \left(\frac{\Gamma}{1-\omega^2}\right)\cos(\omega t) + \mathcal{O}(\varepsilon)
$$

when $\Gamma^2 < 2(1-\omega^2)^2$. Explain why this result breaks down when $\omega = 1/3, 1, 3$.

Solution: Introducing the slow scale $\tau = \varepsilon t$ and substituting the asymptotic expansion

$$
x \sim x_0(t,\tau) + \varepsilon x_1(t,\tau) + \ldots
$$

into the Van der Pol equation we obtain

$$
\begin{aligned}
&\Big[\partial_t + 2\varepsilon\partial_t\partial_\tau + \varepsilon^2\partial_\tau^2\Big]\Big(x_0 + \varepsilon x_1 + \ldots\Big) + \Big(x_0 + \varepsilon x_1 + \ldots\Big) \\
&\quad + \varepsilon\Big[\Big(x_0 + \varepsilon x_1 + \ldots\Big)^2 - 1\Big]\Big[\partial_t + \varepsilon\partial_\tau\Big]\Big(x_0 + \varepsilon x_1 + \ldots\Big) = \Gamma\cos(\omega t).
\end{aligned}
$$

The $\mathcal{O}(1)$ equation is

$$
\Big(\partial_t^2 + 1\Big)x_0 = \Gamma\cos(\omega t)
$$

and its complementary solution is

$$
x_0^c(\tau,t) = A(\tau)\cos(t+\theta(\tau)) = A(\tau)\cos(\Omega(t,\tau)).
$$

Suppose $\omega \neq 1$ so that we prevent secular terms of the $\mathcal{O}(1)$ equation. Assuming a particular solution of the form

$$x_0^p(t,\tau) = B(\tau)\cos(\omega t).$$

Substituting this into the $\mathcal{O}(1)$ equation yields

$$-\omega^2 B + B = \Gamma \implies B = \frac{\Gamma}{1-\omega^2} = \delta.$$

Thus the general solution of the $\mathcal{O}(1)$ equation is

$$x_0(t,\tau) = x_0^c(t,\tau) + x_0^p(t,\tau) = A(\tau)\cos(\Omega(t,\tau)) + \delta\cos(\omega t).$$

The $\mathcal{O}(\varepsilon)$ equation is

$$\begin{aligned}
\left(\partial_t^2 + 1\right)x_1 &= -2\partial_t\partial_\tau x_0 - (x_0^2 - 1)\partial_t x_0 \\
&= 2\Big[A_\tau \sin(\Omega) + C\theta_\tau \cos(\Omega)\Big] - x_0^2\partial_t x_0 + \partial_t x_0 \\
&= 2\Big[A_\tau \sin(\Omega) + C\theta_\tau \cos(\Omega)\Big] - \Big[A\sin(\Omega) + \delta\omega\sin(\omega t)\Big] - x_0^2\partial_t x_0 \\
&= 2A\theta_\tau\cos(\Omega) + \Big[2A_\tau - A\Big]\sin(\Omega) - \delta\omega\sin(\omega t) - x_0^2\partial_t x_0.
\end{aligned}$$

We expand the term $x_0^2\partial_t x_0$ as follows:

$$\begin{aligned}
-x_0^2\partial_t x_0 &= \Big[A^2\cos^2(\Omega) + \delta^2\cos^2(\omega t) + 2A\delta\cos(\Omega)\cos(\omega t)\Big]\Big[A\sin(\Omega) + \delta\omega\sin(\omega t)\Big] \\
&= \frac{A^3}{2}\sin(2\Omega)\cos(\Omega) + A\delta^2\sin(\Omega)\cos^2(\omega t) + A^2\delta\sin(2\Omega)\cos(\omega t) \\
&\quad + A^2\delta\omega\cos^2(\Omega\sin(\omega t) + \frac{\delta^3\omega}{2}\sin(2\omega t)\cos(\omega t) + A\delta^2\omega\cos(\Omega)\sin(2\omega t).
\end{aligned}$$

We carefully apply double-angle formula and product-to-sum identity

$$\begin{aligned}
2\cos^2(X) &= 1 + \cos(2X) \\
2\sin(X)\cos(Y) &= \sin(X+Y) + \sin(X-Y)
\end{aligned}$$

onto each term of $x_0^2\partial_t x_0$:

$$\begin{aligned}
\frac{A^3}{2}\sin(2\Omega)\cos(\Omega) &= \frac{A^3}{4}\Big[\sin(3\Omega) + \sin(\Omega)\Big] \\
A\delta^2\sin(\Omega)\cos^2(\omega t) &= \frac{A\delta^2}{2}\sin(\Omega)\Big[1 + \cos(2\omega t)\Big] \\
&= \frac{A\delta^2}{2}\Big[\sin(\Omega) + \sin(\Omega)\cos(2\omega t)\Big] \\
&= \frac{A\delta^2}{4}\Big[2\sin(\Omega) + \sin(\Omega + 2\omega t) + \sin(\Omega - 2\omega t)\Big]
\end{aligned}$$

$$
\begin{aligned}
A^2\delta\sin(2\Omega)\cos(\omega t) &= \frac{A^2\delta}{2}\Big[\sin(2\Omega+\omega t)+\sin(2\Omega-\omega t)\Big] \\
A^2\delta\omega\cos^2(\Omega)\sin(\omega t) &= \frac{A^2\delta\omega}{2}\sin(\omega t)\Big[1+\cos(2\Omega)\Big] \\
&= \frac{A^2\delta\omega}{2}\Big[\sin(\omega t)+\sin(\omega t)\cos(2\Omega)\Big] \\
&= \frac{A^2\delta\omega}{4}\Big[2\sin(\omega t)+\sin(\omega t+2\Omega)+\sin(\omega t-2\Omega)\Big] \\
&= \frac{A^2\delta\omega}{4}\Big[2\sin(\omega t)+\sin(2\Omega+\omega t)-\sin(2\Omega-\omega t)\Big] \\
\frac{\delta^3\omega}{w}\sin(2\omega t)\cos(\omega t) &= \frac{\delta^3\omega}{4}\Big[\sin(3\omega t)+\sin(\omega t)\Big] \\
A\delta^2\omega\cos(\Omega)\sin(2\omega t) &= \frac{A\delta^2\omega}{2}\Big[\sin(2\omega t+\Omega)+\sin(2\omega t-\Omega)\Big] \\
&= \frac{A\delta^2\omega}{2}\Big[\sin(\Omega+2\omega t)-\sin(\Omega-2\omega t)\Big].
\end{aligned}
$$

Combining everything, the $\mathcal{O}(\varepsilon)$ equation takes the form

$$
\begin{aligned}
\Big(\partial_t^2+1\Big)x_1 = \Big[2A\theta_\tau\Big]\cos(\Omega) &+ \left[2A_\tau - A + \frac{A^3}{4}+\frac{A\delta^2}{2}\right]\sin(\Omega) + \left[\frac{A^3}{4}\right]\sin(3\Omega) \\
&+ \left[-\delta\omega + \frac{A^2\delta\omega}{2}+\frac{\delta^3\omega}{4}\right]\sin(\omega t) + \left[\frac{\delta^3\omega}{4}\right]\sin(3\omega t) \\
&+ \left[\frac{A\delta^2}{4}+\frac{A\delta^2\omega}{2}\right]\sin(\Omega+2\omega t) + \left[\frac{A\delta^2}{4}-\frac{A\delta^2\omega}{2}\right]\sin(\Omega-2\omega t) \\
&+ \left[\frac{A^2\delta}{2}+\frac{A^2\delta\omega}{4}\right]\sin(2\Omega+\omega t) + \left[\frac{A^2\delta}{2}-\frac{A^2\delta\omega}{4}\right]\sin(2\Omega-\omega t).
\end{aligned}
$$

Terms of the form $\sin(\omega t)$ and $\sin(3\omega t)$ will be resonant if $\omega = 1, 1/3$. Terms of the form $\sin(\Omega \pm 2\omega t)$ will be resonant if

$$t \pm 2\omega t = (1\pm 2\omega)t = \pm t \iff \omega = 0, 1.$$

Terms of the form $\sin(2\Omega \pm \omega t)$ will be resonant if

$$2t \pm \omega t = (2\pm\omega)t = \pm t \iff \omega = 1, 3.$$

Therefore, if we assume that $\omega \neq 1/3, 1, 3$ then the only resonant terms on the right-hand side of the $\mathcal{O}(\varepsilon)$ equation are those involving $\cos(\Omega)$ and $\sin(\Omega)$ and we require their coefficients to vanish, *i.e.*

$$
\begin{aligned}
2A\theta_\tau = 0 \quad \text{and} \quad 2A_\tau &= A - \frac{A^3}{4} - \frac{A\delta^2}{2} \\
&= A\left(1-\frac{A^2}{4}-\frac{\delta^2}{2}\right)
\end{aligned}
$$

$$= -A\left(\frac{A^2}{4} - C(\delta)\right), \quad \text{where } C(\delta) = 1 - \frac{\delta^2}{2}.$$

We now perform a case analysis:

(a) When $C(\delta) < 0$, that is, $\delta^2 > 2$, A_τ has only one fixed point at $A = 0$ and this is asymptotically stable, *i.e.* $A(\tau) \longrightarrow 0$ as $\tau \longrightarrow \infty$ for any initial conditions $A(0)$. Therefore, the solution is attracted to

$$x(t) = \delta\cos(\omega t) + \mathcal{O}(\varepsilon) = \left(\frac{\Gamma}{1-\omega^2}\right)\cos(\omega t) + \mathcal{O}(\varepsilon).$$

(b) When $C(\delta) > 0$, that is, $\delta^2 < 2$, A_τ has three fixed points $A_0 = 0, \pm 2\sqrt{C(\delta)}$ and a phase-plane analysis shows that the fixed point $A_0 = 0$ becomes unstable and the other two are stable. Therefore, as $\tau \longrightarrow \infty$ we have

$$A(\tau) \longrightarrow \begin{cases} 2\sqrt{C(\delta)} & \text{if } A(0) > 0, \\ -2\sqrt{C(\delta)} & \text{if } A(0) < 0. \end{cases}$$

In the case of positive $A(0)$, the solution is then attracted to

$$\begin{aligned} x(t) &= 2\sqrt{C(\delta)}\cos(t+\theta_0) + \delta\cos(\omega t) + \mathcal{O}(\varepsilon) \\ &= 2\left[1 - \frac{\Gamma^2}{2(1-\omega^2)^2}\right]^{1/2} + \left(\frac{\Gamma}{1-\omega^2}\right)\cos(\omega t) + \mathcal{O}(\varepsilon). \end{aligned}$$

6. **Multiple scales with nonlinear wave equations.** The Korteweg-de Vries (KdV) equation is

$$u_t + u_x + \alpha u u_x + \beta u_{xxxx} = 0, \quad \boldsymbol{x} \in \mathbb{R}, \quad t > 0,$$

where α, β are positive real constants and $u(x,0) = \varepsilon f(x)$ for $0 < \varepsilon \ll 1$.

(a) Let $\theta = kx - \omega t$ and seek traveling wave solutions using an expansion of the form

$$u(x,t) \sim \varepsilon[u_0(\theta) + \varepsilon u_1(\theta) + \ldots],$$

where $\omega = k - \beta k^3$ and $k > 0$ is a constant. Show that this can lead to secular terms.

Solution:

(b) Use multiple scales (variables $\theta, \varepsilon x, \varepsilon t$) to eliminate the secular terms in part (a) and find a first-term expansion. In the process, show that $f(x)$ must have the form

$$f(x) = A\cos(kx + \phi)$$

for constants A, B, ϕ in order to generate a traveling wave? *Hint: Use the fact that $f(x)$ is independent of ε.*

Solution:

Chapter 4

The Wentzel-Kramers-Brillouin (WKB) Method

The WKB method, named after Wentzel, Kramers and Brillouin, is a method for finding approximate solutions to linear differential equations with spatially varying coefficients. The origin of WKB theory dates back to 1920s where it was developed by Wentzel, Kramers and Brillouin to study time-independent Schrodinger equation. This often arises from the following problem:

$$\frac{d^2y}{dx^2} - q(\varepsilon x)y = 0,$$

with the slowly varying potential energy. To handle such problem, the WKB method introduces an ansatz of the expansion term as a product of slowly varying and exponenetially rapidly varying terms.

4.1 Introductory Example

Consider the differential equation

$$\varepsilon^2 y'' - q(x)y = 0 \quad \text{on } x \in [0,1], \tag{4.1.1}$$

where q is a smooth function. For constant q, the general solution of (4.1.1) is

$$y(x) = a_0 e^{-x\sqrt{q}/\varepsilon} + b_0 e^{x\sqrt{q}/\varepsilon}$$

and the solution either blows up ($q > 0$) or oscillates ($q < 0$) rapidly on a scale of $\mathcal{O}(\varepsilon)$. The hypothesis of the WKB method is that this exponential solution can be generalised to obtain an approximate solution of the full problem (4.1.1).

We start with the following general WKB ansatz:

$$y(x) \sim e^{\theta(x)/\varepsilon^\alpha}\left[y_0(x) + \varepsilon^\alpha y_1(x) + \dots\right] \quad \text{as } \varepsilon \longrightarrow 0 \tag{4.1.2}$$

for some $\alpha > 0$. Here, we assume that the solution varies exponentially with respect to the fast variation. From (4.1.2) we obtain:

$$y' \sim \left\{\varepsilon^{-\alpha}\theta_x y_0 + y_0' + \theta_x y_1 + \dots\right\} e^{\theta/\varepsilon^\alpha} \tag{4.1.3a}$$

$$y'' \sim \left\{\varepsilon^{-2\alpha}\theta_x^2 y_0 + \varepsilon^{-\alpha}\left(\theta_{xx}y_0 + 2\theta_x y_0' + \theta_x^2 y_1\right) + \dots\right\} e^{\theta/\varepsilon^\alpha} \tag{4.1.3b}$$

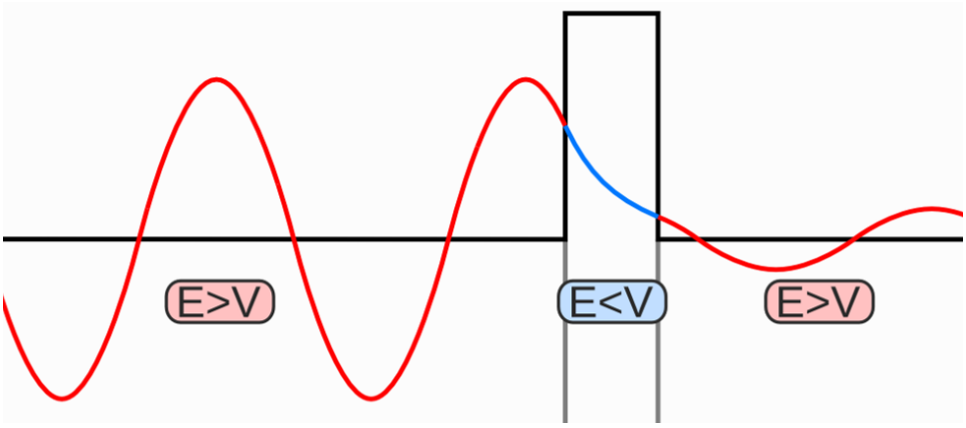


Figure 4.1: An example of turning points: quantum tunneling. Depending on effective potential energy, the solutions have different behavior and need to be matched (Taken from Wikipedia Commons).

$$y''' \sim \left\{\varepsilon^{-3\alpha}\theta_x^3 y_0 + \varepsilon^{-2\alpha}\theta_x\left(3\theta_x y_0' + 3\theta_{xx}y_0 + \theta_x^2 y_1\right) + \dots\right\} e^{\theta/\varepsilon^\alpha} \tag{4.1.3c}$$
$$y'''' \sim \left\{\varepsilon^{-4\alpha}\theta_x^4 y_0 + \varepsilon^{-3\alpha}\theta_x^2\left(6\theta_{xx} y_0 + 4\theta_x y_0' + \theta_x^2 y_1\right) + \dots\right\} e^{\theta/\varepsilon^\alpha} \tag{4.1.3d}$$

Substituting both (4.1.2) and (4.1.3) into (4.1.1) and cancelling the exponential term yield

$$\varepsilon^2\left[\frac{\theta_x^2 y_0}{\varepsilon^{2\alpha}} + \frac{1}{\varepsilon^\alpha}\left(\theta_{xx}y_0 + 2\theta_x y_0' + \theta_x^2 y_1\right) + \dots\right] - q(x)\left[y_0 + \varepsilon^\alpha y_1 + \dots\right] = 0. \tag{4.1.4}$$

Such cancellation is possible due to the linearity of the equation!

Balancing leading-order terms in (4.1.4) we see that $\alpha = 1$. The $\mathcal{O}(1)$ equation is the well-known **eikonal equation**:

$$\theta_x^2 = q(x), \tag{4.1.5}$$

and its solutions (in one-dimensional) are

$$\theta(x) = \pm \int^x \sqrt{q(s)}\, ds. \tag{4.1.6}$$

To determine $y_0(x)$, we need to solve the $\mathcal{O}(\varepsilon)$ equation which is the **transport equation**:

$$\theta_{xx}y_0 + 2\theta_x y_0' + \theta_x^2 y_1 = q(x)y_1. \tag{4.1.7}$$

The y_1 terms cancel out due to the eikonal equation (4.1.5) and (4.1.7) reduces to

$$\theta_{xx}y_0 + 2\theta_x y_0' = 0. \tag{4.1.8}$$

This can be easily solved since it is separable:

$$\begin{aligned}\frac{y_0'}{y_0} &= -\frac{\theta_{xx}}{2\theta_x}\\ \ln|y_0| &= -\frac{1}{2}\ln|\theta_x| + C\\ \ln|y_0| &= -\ln\sqrt{|\theta_x|} + C\end{aligned}$$

$$y_0(x) = \frac{C}{\sqrt{\theta_x}} = Cq(x)^{-1/4},$$

where C is an arbitrary nonzero constant and the last line follows from (4.1.6). Hence, a first-term asymptotic approximation of the general solution of (4.1.1) is

$$y(x) \sim q(x)^{-1/4}\left[a_0 \exp\left(-\frac{1}{\varepsilon}\int^x \sqrt{q(s)}\,ds\right) + b_0 \exp\left(\frac{1}{\varepsilon}\int^x \sqrt{q(s)}\,ds\right)\right], \tag{4.1.9}$$

where a_0, b_0 are arbitrary constants, possibly complex. It is evident that (4.1.9) is valid if $q(x) \neq 0$ on $[0,1]$. The x-values where $q(x) = 0$ are called **turning points** and this nontrivial issue will be addressed in Section 4.2.

Example 4.1.1. Choose $q(x) = -e^{2x}$. Then the WKB approximation (4.1.9) is

$$y(x) \sim e^{-x/2}\left[a_0 e^{-ie^x/\varepsilon} + b_0 e^{ie^x/\varepsilon}\right] = e^{-x/2}\left[\alpha_0\cos(\lambda e^x) + \beta_0\sin(\lambda e^x)\right],$$

where $\lambda = 1/\varepsilon$. With boundary conditions $y(0) = a, y(1) = b$, we obtain

$$y(x) \sim e^{-x/2}\left(\frac{b\sqrt{e}\sin\left(\lambda(e^x - 1)\right) - a\sin\left(\lambda(e^x - e)\right)}{\sin\left(\lambda(e-1)\right)}\right).$$

The exact solution of (4.1.1) with $q(x) = -e^{2x}$ can be solved as follows. Performing a change of variable $\tilde{x} = e^x/\varepsilon = \lambda e^x$, we obtain

$$x = \ln(\varepsilon) + \ln(\tilde{x}) \implies \frac{dx}{d\tilde{x}} = \frac{1}{\tilde{x}}.$$

Setting $Y(\tilde{x}) = y(x)$, it follows from the chain rule that

$$\begin{aligned} \frac{dY}{d\tilde{x}} &= \frac{dy}{dx}\frac{dx}{d\tilde{x}} &&= \frac{y'}{\tilde{x}} \\ \frac{d^2Y}{d\tilde{x}^2} &= -\frac{y'}{\tilde{x}^2} + \frac{y''}{\tilde{x}^2} &&= -\frac{1}{\tilde{x}}\frac{dY}{d\tilde{x}} + \frac{y''}{\tilde{x}^2}. \end{aligned}$$

Consequently, the equation of $Y(\tilde{x})$ is the zeroth-order Bessel's differential equation

$$\tilde{x}^2\frac{d^2Y}{d\tilde{x}^2} + \tilde{x}\frac{dY}{d\tilde{x}} + \tilde{x}^2 Y = 0,$$

and the solution of this is

$$Y(\tilde{x}) = c_0 J_0(\tilde{x}) + d_0 Y_0(\tilde{x}) = c_0 J_0(\lambda e^x) + d_0 Y_0(\lambda e^x) = y(x),$$

where $J_0(\cdot)$ and $Y_0(\cdot)$ are the zeroth-order Bessel functions of the first and second kinds respectively. Finally, solving for c_0 and d_0 using the boundary conditions yields

$$\begin{aligned} c_0 &= \frac{1}{D}\left[bY_0(\lambda) - aY_0(\lambda e)\right] \\ d_0 &= \frac{1}{D}\left[aJ_0(\lambda e) - bJ_0(\lambda)\right] \\ D &= J_0(\lambda e)Y_0(\lambda) - Y_0(\lambda e)J_0(\lambda). \end{aligned}$$

One can plot the exact solution and the WKB approximation and see that their difference is almost zero!

To measure the error of the WKB approximation (4.1.9), we look at the $\mathcal{O}(\varepsilon^2)$ equation which has the form

$$\theta_{xx}y_1 + 2\theta_x y_1' + \theta_x^2 y_2 + y_0'' = q(x)y_2. \tag{4.1.10}$$

The y_2 terms vanish due to the eikonal equation (4.1.5) and so (4.1.10) reduces to

$$\theta_{xx}y_1 + 2\theta_x y_1' + y_0'' = 0. \tag{4.1.11}$$

Because the first two terms of (4.1.11) are similar to the transport equation (4.1.8), we make an ansatz $y_1(x) = y_0(x)w(x)$. (4.1.11) reduces to

$$2\theta_x y_0 w' + y_0'' = 0. \tag{4.1.12}$$

Suppose $q(x) > 0$ so that θ_x is a real-valued function. Rearranging (4.1.12) in terms of w' and integrating by parts with respect to x we obtain

$$\begin{aligned}
2\theta_x y_0 w' &= -y_0'' \\
\frac{2C\theta_x w'}{\sqrt{\theta_x}} &= -\frac{d^2}{dx^2}\left(\frac{C}{\sqrt{\theta_x}}\right) = \frac{d}{dx}\left(\frac{C\theta_{xx}}{2\theta_x^{3/2}}\right) \\
w' &= \frac{1}{4}\frac{d}{dx}\left(\frac{\theta_{xx}}{\theta_x^{3/2}}\right)\left(\frac{1}{\sqrt{\theta_x}}\right) \\
w(x) &= \frac{1}{4}\int^x \left[\frac{d}{dx}\left(\frac{\theta_{xx}}{\theta_x^{3/2}}\right)\right]\left(\frac{1}{\sqrt{\theta_x}}\right) ds \\
&= d + \frac{1}{4}\left(\frac{\theta_{xx}}{\theta_x^2}\right) - \frac{1}{4}\int^x \left(\frac{\theta_{xx}}{\theta_x^{3/2}}\right)\frac{d}{dx}\left(\frac{1}{\sqrt{\theta_x}}\right) ds \\
&= d + \frac{1}{4}\left(\frac{\theta_{xx}}{\theta_x^2}\right) + \frac{1}{8}\int^x \left(\frac{\theta_{xx}^2}{\theta_x^3}\right) ds,
\end{aligned}$$

where d is an arbitrary constant. On the other hand, θ_x is a complex-valued function if $q(x) < 0$, *i.e.* $\theta_x = \pm i\sqrt{-q}$. We then have

$$\begin{aligned}
\theta_{xx} &= \pm\frac{i}{2}\left(\frac{-q_x}{\sqrt{-q}}\right) &&= \mp\frac{iq_x}{2\sqrt{-q}} \\
\frac{\theta_{xx}}{\theta_x^2} &= \mp\frac{iq_x}{2q\sqrt{-q}} &&= \pm\frac{iq_x}{2(-q)^{3/2}} \\
\theta_{xx}^2 &= \frac{-q_x^2}{4(-q)} &&= \frac{q_x^2}{4q} \\
\theta_x^3 &= (\pm i)^3\left(\sqrt{-q}\right)^3 &&= \mp i\,(-q)^{3/2} \\
\frac{\theta_{xx}^2}{\theta_x^3} &= \frac{q_x^2}{\mp 4iq(-q)^{3/2}} &&= \mp\frac{iq_x^2}{4(-q)^{5/2}}.
\end{aligned}$$

Consequently,

$$w(x) = \begin{cases} d + \dfrac{1}{8}\dfrac{q_x}{q^{3/2}} + \dfrac{1}{32}\displaystyle\int^x \left(\dfrac{q_x^2}{q^{5/2}}\right) ds & \text{if } \theta_x(x) = \sqrt{q(x)}, \\ d - \dfrac{1}{8}\dfrac{q_x}{q^{3/2}} - \dfrac{1}{32}\displaystyle\int^x \left(\dfrac{q_x^2}{q^{5/2}}\right) ds & \text{if } \theta_x(x) = -\sqrt{q(x)}, \\ d + \dfrac{1}{8}\dfrac{iq_x}{(-q)^{3/2}} - \dfrac{1}{32}\displaystyle\int^x \left(\dfrac{iq_x^2}{(-q)^{5/2}}\right) ds & \text{if } \theta_x(x) = i\sqrt{-q(x)}, \\ d - \dfrac{1}{8}\dfrac{iq_x}{(-q)^{3/2}} + \dfrac{1}{32}\displaystyle\int^x \left(\dfrac{iq_x^2}{(-q)^{5/2}}\right) ds & \text{if } \theta_x(x) = -i\sqrt{-q(x)}. \end{cases}$$

Finally, for small ε the WKB ansatz (4.1.2) is well-ordered provided

$$|\varepsilon y_1(x)| \ll |y_0(x)|, \quad \text{or} \quad |\varepsilon w(x)| \ll 1.$$

In terms of the function $q(x)$ and its first derivatives, for $x \in [x_0, x_1]$ we will have an accurate approximation if

$$\varepsilon \left[|d| + \frac{1}{32}\left|\frac{q_x}{q^{3/2}}\right| \left(4 + \int_{x_0}^{x_1} \left|\frac{q_x}{q}\right| dx\right)\right] \ll 1,$$

where $|\cdot| := \|\cdot\|_\infty$ over the interval $[x_0, x_1]$. We stress that this condition holds if the interval $[x_0, x_1]$ does not contain a turning point.

Remark 4.1.2. The constants a_0, b_0 in (4.1.9) and d in $w(x)$ are determined from boundary conditions. However, it is very possible that these constants depend on ε. It is therefore necessary to make sure this dependence does not interfere with the ordering assumed in the WKB ansatz (4.1.2).

4.2 Turning Points

This section is devoted to the analysis of turning points of $q(x)$. Assume $q(x)$ is smooth and has a simple zero at $x_t \in [0, 1]$, *i.e.* $q(x_t) = 0$ and $q'(x_t) \neq 0$. For concreteness, we take $q'(x_t) > 0$ and so we expect solutions of (4.1.1) to be oscillatory for $x < x_t$ and exponential for $x > x_t$. We can apply the WKB method on the regions $\{x < x_t\}$ and $\{x > x_t\}$. More precisely, from (4.1.9) we have

$$y \sim \begin{cases} y_L(x, x_t) & \text{if } x < x_t, \\ y_R(x, x_t) & \text{if } x > x_t, \end{cases} \tag{4.2.1}$$

where

$$y_L(x, x_t) = \frac{1}{q(x)^{1/4}}\left[a_L \exp\left(-\frac{1}{\varepsilon}\int_x^{x_t} \sqrt{q(s)}\, ds\right) + b_L \exp\left(\frac{1}{\varepsilon}\int_x^{x_t} \sqrt{q(s)}\, ds\right)\right] \tag{4.2.2a}$$

$$y_R(x, x_t) = \frac{1}{q(x)^{1/4}}\left[a_R \exp\left(-\frac{1}{\varepsilon}\int_{x_t}^{x} \sqrt{q(s)}\, ds\right) + b_R \exp\left(\frac{1}{\varepsilon}\int_{x_t}^{x} \sqrt{q(s)}\, ds\right)\right]. \tag{4.2.2b}$$

An important realization is that these coefficients a_L, b_L, a_R, b_R are not all independent. In addition to the two boundary conditions at $x = 0$ and $x = 1$, we also have matching conditions in a transition layer centered at $x = x_t$.

4.2.1 Transition layer

Following the boundary layer analysis, we introduce the *boundary layer coordinate*

$$\tilde{x} = \frac{x - x_t}{\varepsilon^\beta} \quad \text{or} \quad x = x_t + \varepsilon^\beta \tilde{x}.$$

We can reduce (4.1.1) by expanding the function $q(x)$ around the turning point x_t

$$\begin{aligned} q(x) = q(x_t + \varepsilon^\beta \tilde{x}) &= q(x_t) + q'(x_t)\varepsilon^\beta \tilde{x} + \dots \\ &\approx \varepsilon^\beta \tilde{x} q'(x_t). \end{aligned}$$

Denote the inner solution by $Y(\tilde{x})$. Transforming (4.1.1) using

$$\frac{d}{dx} = \frac{1}{\varepsilon^\beta}\frac{d}{d\tilde{x}}$$

gives the inner equation

$$\varepsilon^{2-2\beta} Y'' - \left(\varepsilon^\beta \tilde{x} q'_t + \dots\right) Y = 0, \tag{4.2.3}$$

where $q'_t := q'(x_t)$. Balancing leading-order terms in (4.2.3) means we require

$$2 - 2\beta = \beta \implies \beta = \frac{2}{3}.$$

Since it is not clear what the asymptotic sequence should be, we take the asymptotic expansion to be

$$Y \sim \varepsilon^\gamma Y_0(\tilde{x}) + \dots. \tag{4.2.4}$$

The $\mathcal{O}(\varepsilon^{2/3})$ equation is

$$Y_0'' - \tilde{x} q'_t Y_0 = 0, \quad -\infty < \tilde{x} < \infty. \tag{4.2.5}$$

Performing a coordinate transformation $s = (q'_t)^{1/3}\tilde{x}$, (4.2.5) becomes the **Airy's equation**:

$$\frac{d^2 Y_0}{ds^2} - sY_0 = 0, \quad -\infty < s < \infty, \tag{4.2.6}$$

and this can be solved either using power series expansion or Laplace transform. The general solution of (4.2.6) is

$$Y_0(s) = a\mathrm{Ai}(s) + b\mathrm{Bi}(s), \tag{4.2.7}$$

where $\mathrm{Ai}(\cdot)$ and $\mathrm{Bi}(\cdot)$ are Airy functions of the first and the second kinds respectively. It is well-known that

$$\begin{aligned} \mathrm{Ai}(x) &= \frac{1}{3^{2/3}\pi}\sum_{k=0}^\infty \frac{1}{k!}\Gamma\left(\frac{k+1}{3}\right)\sin\left(\frac{2\pi}{3}(k+1)\right)\left(3^{1/3}x\right)^k \\ &= \mathrm{Ai}(0)\left(1 + \frac{1}{6}x^3 + \dots\right) + \mathrm{Ai}'(0)\left(x + \frac{1}{12}x^4 + \dots\right) \\ \mathrm{Bi}(x) &= e^{i\pi/6}\mathrm{Ai}\left(xe^{2\pi i/3}\right) + e^{-i\pi/6}\mathrm{Ai}\left(xe^{-2\pi i/3}\right) \\ &= \mathrm{Bi}(0)\left(1 + \frac{1}{6}x^3 + \dots\right) + \mathrm{Bi}'(0)\left(x + \frac{1}{12}x^4 + \dots\right), \end{aligned}$$

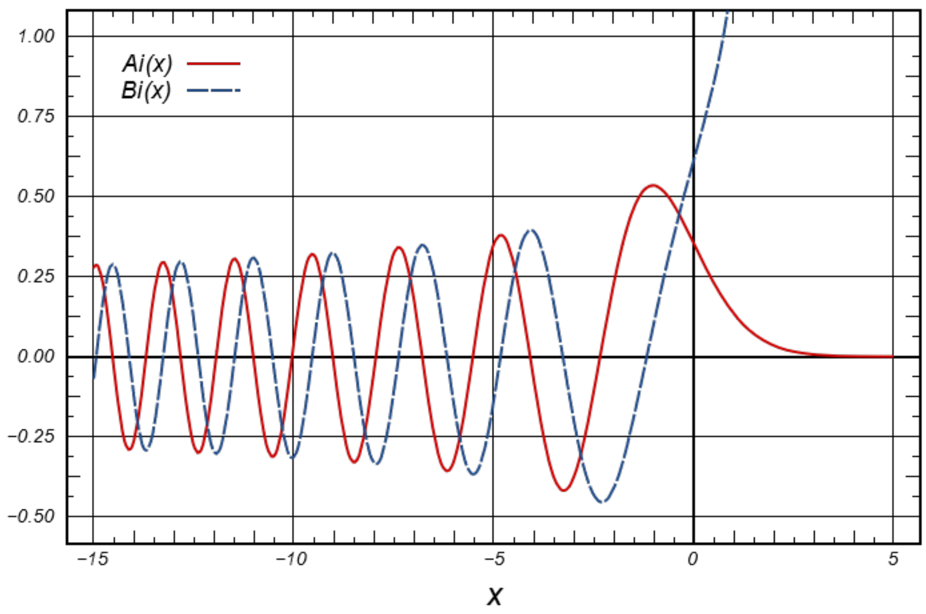


Figure 4.2: Plot of the two Airy functions (Taken from Wikipedia Commons).

where $\Gamma(\cdot)$ is the gamma function. Setting $\xi = \dfrac{2}{3}|x|^{3/2}$, we also have that

$$\mathrm{Ai}(x) \sim \begin{cases} \dfrac{1}{\sqrt{\pi}|x|^{1/4}}\left[\cos\left(\xi - \dfrac{\pi}{4}\right) + \dfrac{5}{72\xi}\sin\left(\xi - \dfrac{\pi}{4}\right)\right] & \text{if } x \longrightarrow -\infty, \\ \dfrac{1}{2\sqrt{\pi}|x|^{1/4}}e^{-\xi}\left[1 - \dfrac{5}{72}\xi\right] & \text{if } x \longrightarrow +\infty, \end{cases} \tag{4.2.8a}$$

$$\mathrm{Bi}(x) \sim \begin{cases} \dfrac{1}{\sqrt{\pi}|x|^{1/4}}\left[\cos\left(\xi + \dfrac{\pi}{4}\right) + \dfrac{5}{72\xi}\sin\left(\xi + \dfrac{\pi}{4}\right)\right] & \text{if } x \longrightarrow -\infty, \\ \dfrac{1}{\sqrt{\pi}|x|^{1/4}}e^{\xi}\left[1 + \dfrac{5}{72}\xi\right] & \text{if } x \longrightarrow +\infty. \end{cases} \tag{4.2.8b}$$

4.2.2 Matching

From (4.2.7), the general solution of (4.1.1) in the transition layer is

$$Y_0(\tilde{x}) = a\mathrm{Ai}\left[(q_t')^{1/3}\,\tilde{x}\right] + b\mathrm{Bi}\left[(q_t')^{1/3}\,\tilde{x}\right]. \tag{4.2.9}$$

We now have 6 undetermined constants from (4.2.2) and (4.2.9), but these are all connected since the inner solution (4.2.9) must match the outer solutions (4.2.2). These will results in two arbitrary constants in the general solution (4.2.1). Since the inner solution is unbounded, we introduce an intermediate variable

$$x_\eta = \frac{x - x_t}{\varepsilon^\eta}, \quad 0 < \eta < \frac{2}{3},$$

where the interval for η comes from the requirement that the scaling for the intermediate variable must lie between the outer scale, $\mathcal{O}(1)$ and the inner scale, $\mathcal{O}(\varepsilon^{2/3})$.

4.2.3 Matching for $x > x_t$

We first change the stretched variable $\tilde{x}$ to the intermediate variable x_η:

$$\tilde{x} = \frac{x - x_t}{\varepsilon^\beta} = \frac{x - x_t}{\varepsilon^\eta \varepsilon^{\beta-\eta}} = \varepsilon^{\eta-\beta} x_\eta = \varepsilon^{\eta-2/3} x_\eta.$$

Note that $x_\eta > 0$ since $x > x_t$. From (4.2.4) and (4.2.9), the inner solution $Y(\tilde{x})$ now becomes

$$\begin{aligned} Y &\sim \varepsilon^\gamma Y_0\left(\varepsilon^{\eta-2/3} x_\eta\right) + \dots \\ &\sim \varepsilon^\gamma \left[a\mathrm{Ai}\left((q_t')^{1/3}\varepsilon^{\eta-2/3}x_\eta\right) + b\mathrm{Bi}\left((q_t')^{1/3}\varepsilon^{\eta-2/3}x_\eta\right)\right] + \dots \\ &\sim \varepsilon^\gamma \left[a\mathrm{Ai}(r) + b\mathrm{Bi}(r)\right] + \dots \\ &\sim \varepsilon^\gamma \left[\frac{a}{2\sqrt{\pi} r^{1/4}} \exp\left(-\frac{2}{3} r^{3/2}\right) + \frac{b}{\sqrt{\pi} r^{1/4}} \exp\left(\frac{2}{3} r^{3/2}\right)\right], \end{aligned} \tag{4.2.10}$$

where $r = q'(x_t)^{1/3}\varepsilon^{\eta-2/3}x_\eta > 0$ and the last line follows from (4.2.8). On the other hand, since

$$\begin{aligned} \int_{x_t}^{x} \sqrt{q(s)}\, ds &\sim \int_{x_t}^{x_t+\varepsilon^\eta x_\eta} \sqrt{(s - x_t)q_t'}\, ds \\ &= \sqrt{q_t'}\left[\frac{2}{3}(s - x_t)^{3/2}\right]\Bigg|_{x_t}^{x_t+\varepsilon^\eta x_\eta} \\ &= \frac{2}{3}\sqrt{q_t'}\,(\varepsilon^\eta x_\eta)^{3/2} \\ &= \frac{2}{3}\varepsilon r^{3/2} \end{aligned}$$

and

$$q(x)^{-1/4} \sim \left[q(x_t) + (x - x_t)q_t'\right]^{-1/4} = \left[\varepsilon^\eta x_\eta q_t'\right]^{-1/4} = \varepsilon^{-1/6}\left(q_t'\right)^{-1/6} r^{-1/4},$$

the right outer solution y_R becomes

$$y_R \sim \frac{\varepsilon^{-1/6}}{(q_t')^{1/6}\, r^{1/4}}\left[a_R \exp\left(-\frac{2}{3} r^{3/2}\right) + b_R \exp\left(\frac{2}{3} r^{3/2}\right)\right]. \tag{4.2.11}$$

Consequently, matching (4.2.10) the right outer solution y_R with (4.2.11) the inner solution Y yields the following:

$$\gamma = -\frac{1}{6}, \quad a_R = \frac{a}{2\sqrt{\pi}}\left(q_t'\right)^{1/6}, \quad b_R = \frac{b}{\sqrt{\pi}}\left(q_t'\right)^{1/6}. \tag{4.2.12}$$

4.2.4 Matching for $x < x_t$

Because $x < x_t$, we have $x_\eta < 0$ which introduces complex numbers into the outer solution y_L. Using the asymptotic properties of Airy functions as $r \longrightarrow -\infty$ (see (4.2.8)), the inner solution becomes

$$Y \sim \varepsilon^\gamma \left[a\mathrm{Ai}(r) + b\mathrm{Bi}(r)\right] + \dots$$

$$\sim \varepsilon^{\gamma}\left[\frac{a}{\sqrt{\pi}|r|^{1/4}}\cos\left(\frac{2}{3}|r|^{3/2}-\frac{\pi}{4}\right)+\frac{b}{\sqrt{\pi}|r|^{1/4}}\cos\left(\frac{2}{3}|r|^{3/2}+\frac{\pi}{4}\right)\right].$$

Using the identity $\cos\theta = (e^{i\theta}+e^{-i\theta})/2$, a more useful form of the inner expansion Y as $r \longrightarrow -\infty$ is

$$Y \sim \frac{\varepsilon^{\gamma}}{2\sqrt{\pi}|r|^{1/4}}\Big[\left(ae^{-i\pi/4}+be^{i\pi/4}\right)e^{i\zeta}+\left(ae^{i\pi/4}+be^{-i\pi/4}\right)e^{-i\zeta}\Big], \tag{4.2.13}$$

where $\zeta = \dfrac{2}{3}|r|^{3/2}$. On the other hand, since

$$\begin{aligned}\int_x^{x_t}\sqrt{q(s)}\,ds &\sim \int_{x_t+\varepsilon^{\eta}x_{\eta}}^{x_t}\sqrt{(s-x_t)q_t'}\,ds\\ &= \sqrt{q_t'}\left[\frac{2}{3}(s-x_t)^{3/2}\right]\Bigg|_{x_t+\varepsilon^{\eta}x_{\eta}}^{x_t}\\ &= -\frac{2}{3}\sqrt{q_t'}\left(\varepsilon^{\eta}x_{\eta}\right)^{3/2}\\ &= -\frac{2}{3}\varepsilon|r|^{3/2}\left(-1\right)^{3/2}\\ &= \frac{2}{3}i\varepsilon|r|^{3/2},\end{aligned}$$

and

$$\begin{aligned}q(x)^{-1/4} \sim \left[\varepsilon^{\eta}x_{\eta}q_t'\right]^{-1/4} &= \varepsilon^{-1/6}\left(q_t'\right)^{-1/6}|r|^{-1/4}\left(-1\right)^{-1/4}\\ &= \varepsilon^{-1/6}\left(q_t'\right)^{-1/6}|r|^{-1/4}e^{-i\pi/4},\end{aligned}$$

the left outer solution y_L becomes

$$y_L \sim \frac{\varepsilon^{-1/6}e^{-i\pi/4}}{\left(q_t'\right)^{1/6}|r|^{1/4}}\left[a_Le^{-i\zeta}+b_Le^{i\zeta}\right]. \tag{4.2.14}$$

Consequently, matching (4.2.14) the left outer solution y_L with (4.2.13) the inner solution Y yields the following:

$$a_L = \frac{\left(q_t'\right)^{1/6}}{2\sqrt{\pi}}\left(ia+b\right), \quad b_L = \frac{\left(q_t'\right)^{1/6}}{2\sqrt{\pi}}\left(a+ib\right) = i\bar{a}_L. \tag{4.2.15}$$

From (4.2.12), it follows that

$$a_L = ia_R + \frac{b_R}{2}, \quad b_L = a_R + \frac{i}{2}b_R \tag{4.2.16}$$

or in matrix form

$$\begin{bmatrix}a_L\\ b_L\end{bmatrix} = \begin{bmatrix}i & 1/2\\ 1 & i/2\end{bmatrix}\begin{bmatrix}a_R\\ b_R\end{bmatrix}. \tag{4.2.17}$$

4.2.5 Conclusion

Because we assume $q(t) < 0$ for $x < x_t$, this introduces complex numbers on y_L:

$$q(x)^{-1/4} = e^{-i\pi/4}|q(x)|^{-1/4}$$
$$\int_x^{x_t} \sqrt{q(s)}\, ds = i\int_x^{x_t} \sqrt{|q(s)|}\, ds$$

In conclusion, we have

$$y(x) = \begin{cases} y_L(x, x_t) & \text{if } x < x_t, \\ y_R(x, x_t) & \text{if } x > x_t, \end{cases}$$

where

$$\begin{aligned}
y_L(x, x_t) &= \frac{1}{|q(x)|^{1/4}}\left[\left(ia_R + \frac{b_R}{2}\right)e^{-i\theta(x)/\varepsilon}e^{-i\pi/4} + \left(a_R + \frac{ib_R}{2}\right)e^{i\theta(x)/\varepsilon}e^{-i\pi/4}\right] \\
&= \frac{1}{|q(x)|^{1/4}}\left[a_R\left(e^{-i\theta(x)/\varepsilon}e^{i\pi/4} + e^{i\theta(x)/\varepsilon}e^{-i\pi/4}\right) + \frac{b_R}{2}\left(e^{-i\theta(x)/\varepsilon}e^{-i\pi/4} + e^{i\theta(x)/\varepsilon}e^{i\pi/4}\right)\right] \\
&= \frac{1}{|q(x)|^{1/4}}\left[2a_R\cos\left(\frac{1}{\varepsilon}\theta(x) - \frac{\pi}{4}\right) + b_R\cos\left(\frac{1}{\varepsilon}\theta(x) + \frac{\pi}{4}\right)\right] \\
y_R(x, x_t) &= \frac{1}{q(x)^{1/4}}\left[a_R e^{-\kappa(x)/\varepsilon} + b_R e^{\kappa(x)/\varepsilon}\right] \\
\theta(x) &= \int_x^{x_t} \sqrt{|q(s)|}\, ds \\
\kappa(x) &= \int_{x_t}^{x} \sqrt{|q(s)|}\, ds.
\end{aligned}$$

Example 4.2.1. Consider $q(x) = x(2 - x)$, where $-1 < x < 1$. The simple turning point is at $x_t = 0$, with $q'(0) = 2 > 0$. One can compute and show that

$$\begin{aligned}
\theta(x) &= \frac{1}{2}(1 - x)\sqrt{x(x-2)} - \frac{1}{2}\ln\left[1 - x + \sqrt{x(x-2)}\right], \quad x < 0 \\
\kappa(x) &= \frac{1}{2}(x - 1)\sqrt{x(2-x)} - \frac{1}{2}\arcsin(x - 1) + \frac{\pi}{4}, \qquad x > 0.
\end{aligned}$$

4.2.6 The opposite case: $q_t' < 0$

The approximation derived for $q'(x_t) > 0$ can be used when $q'(x_t) < 0$ by simply making the change of variables $z = x_t - x$. This results in

$$\begin{bmatrix} a_L \\ b_L \end{bmatrix} = \begin{bmatrix} i/2 & 1 \\ 1/2 & i \end{bmatrix}\begin{bmatrix} a_R \\ b_R \end{bmatrix}.$$

Consequently,

$$\begin{aligned}
y_L(x) &= \frac{1}{q(x)^{1/4}}\left[a_L e^{\theta(x)/\varepsilon} + b_L e^{-\theta(x)/\varepsilon}\right] \\
y_R(x) &= \frac{1}{|q(x)|^{1/4}}\left[2b_L\cos\left(\frac{1}{\varepsilon}\kappa(x) - \frac{\pi}{4}\right) + a_L\cos\left(\frac{1}{\varepsilon}\kappa(x) + \frac{\pi}{4}\right)\right].
\end{aligned}$$

4.3 Wave Propagation and Energy Methods

In this section, we study how to obtain an asymptotic approximation of a travelling-wave solution of the following PDE which models the string displacement

$$u_{xx} = \mu^2(x)u_{tt} + \alpha(x)u_t + \beta(x)u, \quad 0 < x < \infty, t > 0 \tag{4.3.1a}$$
$$u(0,t) = \cos(\omega t) \tag{4.3.1b}$$

The terms $\alpha(x)u_t$ and βu correspond to damping and elastic support respectively. From the initial condition, we see that the string is periodically forced at the left end and so the solution will develop into a wave that propagates to the right.

Observe that there is no obvious small parameter ε, but we will extract one from the following observation. In the special case where $\alpha = \beta = 0$ and μ equals some constant, (4.3.1) reduces to the classical wave equation and we obtain the right-moving plane waves

$$u(x,t) = e^{i(wt-kx)}, \quad \text{where the wavenumber } k \text{ satisfies } k = \pm\omega\mu.$$

For higher temporal frequencies $\omega \gg 1$, these waves have short wavelength, *i.e.* $\lambda = \left|\frac{2\pi}{k}\right| \ll 1$. Motivated by this, we choose $\varepsilon = 1/\omega$ and construct an asymptotic approximation of the travelling-wave solution of (4.3.1) in the case of a high frequency. The WKB ansatz is assumed to be

$$u(x,t) \sim \exp\left[i\left(wt - \underbrace{w^\gamma\theta(x)}_{\text{fast oscillation}}\right)\right]\left\{u_0(x) + \underbrace{\frac{1}{w^\gamma}u_1(x)}_{\text{slowly-varying}} + \dots\right\}. \tag{4.3.2}$$

Substituting (4.3.2) into (4.3.1) we obtain

$$\begin{aligned}-\omega^{2\gamma}\theta_x^2\left(u_0 + w^{-\gamma}u_1 + \dots\right) + iw^\gamma\theta_x\left(\partial_x u_0 + \dots\right) + \frac{d}{dx}\left(i\omega^\gamma\theta_x u_0 + \dots\right)\\ = -\mu^2\omega^2\left(u_0 + \omega^{-\gamma}u_1 + \dots\right) - i\omega\alpha\left(u_0 + \dots\right) + \beta\left(u_0 + \dots\right).\end{aligned}$$

Balancing the first terms on each side of this equation gives $\gamma = 1$. The $\mathcal{O}(\omega^2) = \mathcal{O}(1/\varepsilon^2)$ equation is the eikonal equation:

$$\theta_x^2 = \mu^2(x), \tag{4.3.3}$$

and its solutions are

$$\theta(x) = \pm\int_0^x \mu(s)\,ds. \tag{4.3.4}$$

We choose the positive solution as we are considering the right-moving waves. The $\mathcal{O}(\omega) = \mathcal{O}(1/\varepsilon)$ equation is the transport equation:

$$-\theta_x^2 u_1 + i\theta_x\partial_x u_0 + i\left(\theta_x\partial_x u_0 + \theta_{xx}u_0\right) = -\mu^2 u_1 - i\alpha u_0. \tag{4.3.5}$$

The u_1 terms cancel out due to the eikonal equation (4.3.3), so (4.3.5) reduces to

$$\theta_{xx}u_0 + 2\theta_x\partial_x u_0 = -\alpha u_0, . \tag{4.3.6}$$

With $\theta_x = \mu(x)$, we can rearrange (4.3.6) and obtain a first order ODE in u_0:

$$\partial_x u_0 + \left(\frac{\mu_x + \alpha}{2\mu}\right)u_0 = 0, \tag{4.3.7}$$

which can be solved using the method of integrating factor. The integrating factor is given by

$$I(x) = \exp\left(\int_0^x \left(\frac{\mu_s(s) + \alpha(s)}{2\mu(s)}\right) ds\right) = \sqrt{\mu(x)}\exp\left(\frac{1}{2}\int_0^x \frac{\alpha(s)}{\mu(s)}\, ds\right),$$

and so (4.3.7) can be written as

$$\frac{d}{dx}\left(I(x)u_0\right) = 0, \quad u_0 = \frac{a_0}{I(x)} = \frac{a_0}{\sqrt{\mu(x)}}\exp\left(-\frac{1}{2}\int_0^x \frac{\alpha(s)}{\mu(s)}\, ds\right). \tag{4.3.8}$$

Finally, imposing the boundary condition at $x = 0$ we obtain a first-term asymptotic expansion of the travelling-wave solution of (4.3.1)

$$u(x,t) \sim \sqrt{\frac{\mu(0)}{\mu(x)}}\exp\left[-\frac{1}{2}\int_0^x \frac{\alpha(s)}{\mu(s)}\, ds\right]\cos\left(\omega t - \omega\int_0^x \mu(s)\, ds\right). \tag{4.3.9}$$

Observe that in (4.3.9) the amplitude and phase of the travelling wave depend on the spatial position x. Interestingly, (4.3.9) is independent of $\beta(x)$.

4.3.1 Connection to energy methods

Energy methods are extremely powerful in the study of wave-related problems. To determine the energy equation in this case, we multiply (4.3.1) by u_t:

$$\begin{aligned} u_t u_{xx} &= \mu^2(x)u_t u_{tt} + \alpha(x)u_t^2 + \beta(x)uu_t \\ \partial_x(u_t u_x) - \frac{1}{2}\partial_t\left(u_x^2\right) &= \frac{1}{2}\mu^2(x)\partial_t\left(u_t^2\right) + \alpha(x)u_t^2 + \beta(x)\partial_t\left(u^2\right) \\ \partial_t\left[\frac{1}{2}\mu^2(x)\left(u_t^2\right) + \frac{1}{2}\beta(x)u^2 + \frac{1}{2}\left(u_x^2\right)\right] - \partial_x\left(u_t u_x\right) &= -\alpha(x)u_t^2 \\ \partial_t E(x,t) + \partial_x S(x,t) &= -\Phi(x,t), \end{aligned}$$

where

$$\begin{aligned} E(x,t) &= \text{energy density} &&:= \frac{1}{2}\mu^2(x)\left(\partial_t u\right)^2 + \frac{1}{2}\left(\partial_x u\right)^2 + \frac{1}{2}\beta(x)u^2 \\ S(x,t) &= \text{energy flux} &&:= -\partial_t u\partial_x u \\ \Phi(x,t) &= \text{dissipation function} &&:= \alpha(x)\left(\partial_t u\right)^2. \end{aligned}$$

We are interested in the energy over some spatial interval of the form $[x_1(t), x_2(t)]$. It follows from Leibniz's rule,

$$\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} E(x,t)\, dx = E(x_2(t),t)\dot{x_2} - E(x_1(t),t)\dot{x_1} + \int_{x_1(t)}^{x_2(t)} \partial_t E(x,t)\, dx \tag{4.3.10a}$$

$$= E(x_2(t),t)\dot{x_2} - E(x_1(t),t)\dot{x_1} - S(x_2(t),t) + S(x_1(t),t) - \int_{x_1(t)}^{x_2(t)} \Phi(x,t)\, dx. \tag{4.3.10b}$$

The term $E(x_j(t),t)\dot{x_j}$ is the change of energy due to the motion of the endpoint, $S(x_j(t),t)$ is the flux of energy across the endpoint due to wave motion and $-\int_{[x_1(t),x_2(t)]}\Phi(x,t)\, dx$ is the energy loss over the interval due to dissipation.

The WKB solution can be written in the more general form:

$$u(x,t) \sim \underbrace{A(x)}_{\text{slowly changing amplitude}} \cos\left[wt - \underbrace{\varphi(x)}_{\text{rapidly changing phase}}\right], \quad \varphi(x) = \omega\theta(x). \tag{4.3.11}$$

It follows that

$$E(x,t) \sim \frac{1}{2}A^2\left(\mu^2\omega^2 + \varphi_x^2\right)\sin^2\left[\omega t - \varphi(x)\right] \tag{4.3.12a}$$
$$S(x,t) \sim \omega\varphi_x A^2 \sin^2\left[\omega t - \varphi(x)\right] \tag{4.3.12b}$$
$$\Phi(x,t) \sim \alpha\omega^2 A^2 \sin^2\left[\omega t - \varphi(x)\right]. \tag{4.3.12c}$$

Note that we neglect A' since A is slowly changing. Suppose we choose $x_i(t)$ satisfying

$$\dot{x}_i = \frac{\omega}{\varphi_x(x_i)} = \text{phase velocity}.$$

Such curves in the $x - t$ plane are called **phase lines**. Then

$$\begin{aligned} E\dot{x} - S &\sim \frac{1}{2}\frac{\omega A^2}{\varphi_x}\left[\mu^2\omega^2 + \varphi_x^2\right]\sin^2\left[\omega t - \varphi(x)\right] - \omega\varphi_x A^2\sin^2\left[\omega t - \varphi(x)\right] \\ &= \frac{1}{2}\frac{\omega A^2}{\varphi_x}\left[\mu^2\omega^2 - \varphi_x^2\right]\sin^2\left[\omega t - \varphi(x)\right] = 0, \end{aligned}$$

since $\theta(x) = \varphi(x)/\omega$ satisfies the eikonal equation (4.3.3). Hence, if $x_2 - x_1 = \mathcal{O}(1/\omega)$ then it follows from (4.3.10) that $\dfrac{dE}{dt} \approx 0$, *i.e.* the total energy remains constant (to the first term) between any two phase lines $x_1(t), x_2(t)$ that are $\mathcal{O}(1/\omega)$ apart.

Recall the energy equation that

$$\partial_t E + \partial_x S = -\Phi.$$

Averaging the energy equation over one period in time results in

$$\partial_x\left(\int_0^{2\pi/\omega} S(x,t)\,dt\right) = -\int_0^{2\pi/\omega}\Phi(x,t)\,dt,$$

where the average of $\partial_t E$ over one period vanishes using (4.3.12) for E. Substituting (4.3.12) for S and Φ, we obtain

$$\begin{aligned} \partial_x\left(\varphi_x A^2\right) &= -\alpha\omega A^2 \\ \partial_x\left(\theta_x A^2\right) &= -\alpha A^2 \\ \theta_{xx}A^2 + 2\theta_x A A_x &= -\alpha A^2 \\ \theta_{xx}A + 2\theta_x A_x &= -\alpha A, \end{aligned}$$

which implies that $A = u_0$ since the last equation is precisely the transport equation (4.3.6). Physically, this means that the transport equation corresponds to the balance of energy over one period in time.

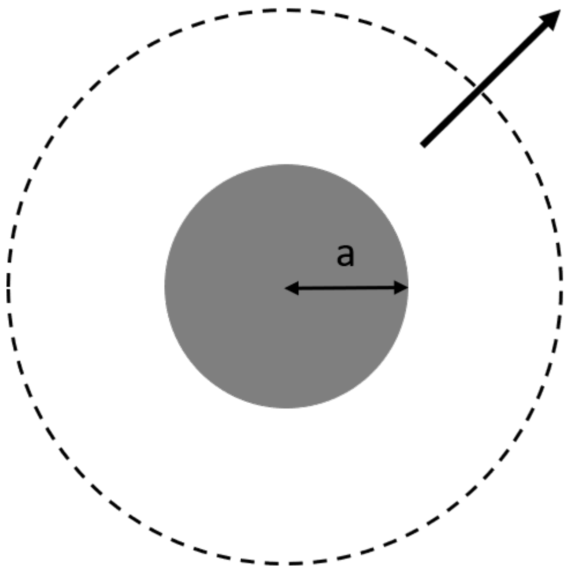


Figure 4.3: Instructive case of the multi-dimensional wave equation. In $\mathbb{R}^2$, the wave propagates from the circle with radius a.

4.4 Higher-Dimensional Waves - Ray Methods

The extension of the WKB method to higher dimensions is relatively straightforward, but the equations could be difficult to solve explicitly. Consider the n-dimensional wave equation

$$\nabla^2 u = \mu^2(\boldsymbol{x})\partial_t^2 u, \quad \boldsymbol{x} \in \mathbb{R}^n, \quad n = 2, 3. \tag{4.4.1}$$

We look for time-harmonic solutions $u(\boldsymbol{x}, t) = e^{-i\omega t}V(\boldsymbol{x})$ and (4.4.1) reduces to the Helmholtz equation

$$\nabla^2 V + \omega^2\mu^2(\boldsymbol{x})V = 0. \tag{4.4.2}$$

It is more instructive to have some understanding of what properties the solution has and how the WKB approximation takes advantage of them. Suppose μ is constant and we want to solve (4.4.2) in the region exterior to the circle $\|\boldsymbol{x}\| = a$ in $\mathbb{R}^2$. Exploiting the geometry leads to the choice of polar coordinates

$$x = \rho\cos(\varphi), \quad y = \rho\sin(\varphi).$$

We impose the Dirichlet boundary condition $V = f(\varphi)$ at $\rho = a$ and the **Sommerfeld radiation condition** which ensures that waves only propagate outward from the circle:

$$\sqrt{\rho}\,[\partial_\rho V - i\omega\mu V] = 0 \quad \text{for } \rho \longrightarrow \infty.$$

Using separation of variables, the general solution of (4.4.2) is given by

$$V(\rho, \varphi) = \sum_{n=-\infty}^{\infty} \alpha_n \left(\frac{H_n^{(1)}(\omega\mu\rho)}{H_n^{(1)}(\omega\mu a)}\right) e^{-in\varphi}, \tag{4.4.3}$$

where $H_n^{(1)}$ is the Hankel function of first kind and the α_n are determined from the boundary condition at $\rho = a$. It is known that for large values of z

$$H_n^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} \exp\left(i\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right).$$

Consequently, in the regime of higher frequency $\omega \gg 1$ (4.4.3) reduces to

$$V(\rho,\varphi) \sim f(\varphi)\sqrt{\frac{a}{\rho}}e^{i\omega\mu(\rho-a)}. \tag{4.4.4}$$

Thus we have a WKB-like solution for constant μ. Radial lines in this example correspond to **rays** and from (4.4.4) we see that along a ray (*i.e.* φ is fixed), the solution has a highly oscillatory component that is multiplied by a slowly varying amplitude $V_0 = f(\varphi)\sqrt{a/\rho}$ that decays as ρ increases.

4.4.1 WKB expansion

We first specify the domain and boundary conditions. The Helmholtz equation (4.4.2) is to be solved in a region exterior to a smooth surface S, where S encloses a bounded convex domain. This means that there is a well-defined unit outward normal at every point on the surface. We impose the Dirichlet boundary condition

$$V(\boldsymbol{x}_0) = f(\boldsymbol{x}_0) \quad \text{for } \boldsymbol{x}_0 \in S$$

and focus only on outward propagating waves.

For higher frequency waves, we take a WKB ansatz of the form

$$V(\boldsymbol{x}) \sim e^{i\omega\theta(\boldsymbol{x})}\left[V_0(\boldsymbol{x}) + \frac{1}{\omega}V_1(\boldsymbol{x}) + \dots\right]. \tag{4.4.5}$$

Then

$$\nabla V \sim \{i\omega\nabla\theta V_0 + i\nabla\theta V_1 + \nabla V_1 + \dots\}\, e^{i\omega\theta} \tag{4.4.6a}$$

$$\nabla^2 V \sim \left\{-\omega^2\nabla\theta\cdot\nabla\theta V_0 + \omega\left(-\nabla\theta\cdot\nabla\theta V_1 + 2i\nabla\theta\cdot\nabla V_0 + \nabla^2\theta V_0\right) + \dots\right\} e^{i\omega\theta}. \tag{4.4.6b}$$

Substituting (4.4.6) into (4.4.2) and rearranging we find that

$$\omega^2\left(-\nabla\theta\cdot\nabla\theta V_0 + \mu^2 V_0\right) + \omega\left[-\nabla\theta\cdot\nabla\theta V_1 + 2i\nabla\theta\cdot\nabla V_0 + i\nabla^2\theta V_0 + \mu^2 V_1\right] + \mathcal{O}(1) = 0$$

$$\left(\nabla\theta\cdot\nabla\theta - \mu^2\right)V_0 + \frac{1}{\omega}\left[\left(\nabla\theta\cdot\nabla\theta - \mu^2\right)V_1 - i\nabla^2\theta V_0 - 2i\nabla\theta\cdot\nabla V_0\right] + \mathcal{O}\left(\frac{1}{\omega^2}\right) = 0.$$

The $\mathcal{O}(1)$ equation is the eikonal equation which is now nontrivial to solve:

$$\nabla\theta\cdot\nabla\theta = \mu^2. \tag{4.4.7}$$

After cancelling the V_1 term using the eikonal equation (4.4.7), the $\mathcal{O}(1/\omega)$ equation is the transport equation

$$2\nabla\theta\cdot\nabla V_0 + \left(\nabla^2\theta\right)V_0 = 0. \tag{4.4.8}$$

Both $\pm\theta$ are solutions to the eikonal equation and we choose the positive solution $+\theta$ since this corresponds to the outward propagating waves.

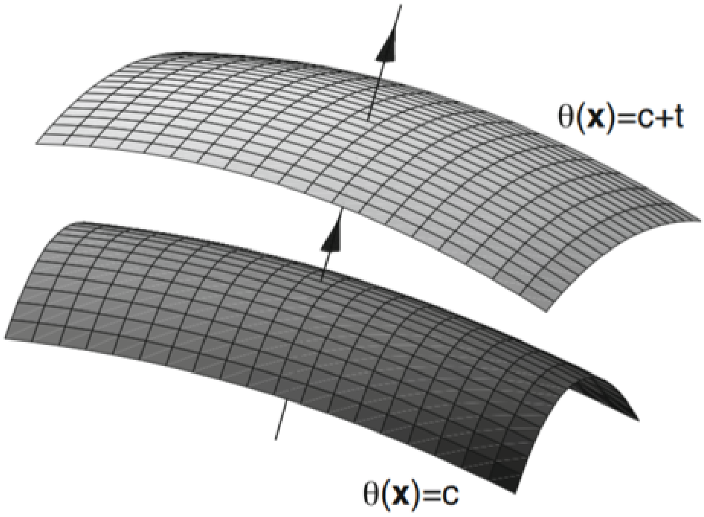


Figure 4.4: Schematic figure of wave fronts in $\mathbb{R}^3$ and the path followed by one of the points in the wave front (Taken from [Hol12, page 267]).

4.4.2 Surfaces and wave fronts

The usual method method for solving the nonlinear eikonal equation (4.4.7) is to introduce **characteristic coordinates**. More precisely, we use curves that are orthogonal to the level surfaces of $\theta(\boldsymbol{x})$ which are also known as **wave fronts** or **phase fronts**.

First, note that the WKB approximation of (4.4.1) has the form

$$u(\boldsymbol{x}, t) \sim e^{i(\omega\theta(\boldsymbol{x})-\omega t)} V_0(\boldsymbol{x}).$$

We introduce the phase function

$$\Theta(\boldsymbol{x}, t) = \omega\theta(\boldsymbol{x}) - \omega t.$$

Suppose we start at $t = 0$ with the surface $S_c = \{\theta(\boldsymbol{x}) = c\}$, so that

$$\Theta(\boldsymbol{x}, 0) = \omega c.$$

As t increases, the points where $\Theta = \omega c$ change, and therefore points forming S_c move and form a new surface $S_{c+t} = \{\theta(\boldsymbol{x}) = c + t\}$. We still have

$$\Theta(\boldsymbol{x}, t) = \omega c.$$

The path each point takes to get from $S_c$to S_{c+t} is obtained from the solution of the eikonal equation and in the WKB method these paths are called **rays**.

The evolution of the wave front generates a natural coordinate system (s, α, β) where α, β comes from parameterising the wave front and s from parameterising the rays. Note that these coordinates are not unique as there are no unique parameterisation for the surfaces and rays. It turns out that determining these coordinates is crucial in the derivation of the WKB approximation.

Example 4.4.1. Suppose we know a-priori that $\theta(\boldsymbol{x}) = \boldsymbol{x}\cdot\boldsymbol{x}$. In this case, the surface S_{c+t} is described by the equation $|\boldsymbol{x}|^2 = c+t$, which is just the sphere with radius $c+t$. The rays are now radial lines and so the points forming S_c move along radial lines to form the surface S_{c+t}. To this end, we use a modified version of spherical coordinates:

$$(x, y, z) = \rho(s)\left(\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha\right),$$

with

$$0 \le \alpha < \pi, \quad 0 \le \beta \le 2\pi, \quad 0 \le s.$$

The function $\rho(s)$ is required to be smooth and strictly increasing. Examples are $\rho = s$, $\rho = e^s - 1$ or $\rho = \ln(1+s)$.

An important property of the preceeding modified spherical coordinates is that (s,α,β) forms an orthogonal coordinate system. That is, under the change of variables $\boldsymbol{x} = \boldsymbol{X}(s,\alpha,\beta)$, the vector $\partial_s\boldsymbol{X}$ tangent to the ray is orthogonal to the wave front S_{c+t}. We now in the opposite case: we need to find $\theta(\boldsymbol{x})$ given conditions on the map $\boldsymbol{X}(s,\alpha,\beta)$. Observe the degree of freedom on specifiy $\boldsymbol{X}$.

4.4.3 Solution of the eikonal equation

In what follows, we will assume that (s,α,β) forms an orthogonal coordinate system. This means that a ray's tangent vector $\partial_s\boldsymbol{X}$ points in the same direction as $\nabla\theta$ when $\boldsymbol{x} = \boldsymbol{X}(s,\alpha,\beta)$, or equivalently

$$\frac{\partial\boldsymbol{X}}{\partial s} = \lambda\nabla\theta, \tag{4.4.9}$$

where λ is a smooth positive function, to be specified later. WLOG, we assume that the rays are parameterised so that $s \ge 0$. One should not confuse s with the arclength parameterisation.

Along a ray,

$$\partial_s\theta(\boldsymbol{X}) = \nabla\theta\cdot\partial_s\boldsymbol{X} = \lambda\nabla\theta\cdot\nabla\theta.$$

Therefore we can rewrite the eikonal equation as

$$\partial_s\theta = \lambda\mu^2 \tag{4.4.10}$$

which can be integrated directly to yield

$$\theta(s,\alpha,\beta) = \theta(0,\alpha,\beta) + \int_0^s \lambda\mu^2\,d\sigma, \tag{4.4.11}$$

assuming we can find such a coordinate system (s,α,β). This amounts to solving (4.4.9) which is generally nonlinear and requires the assistance of numerical method. Nonetheless, we still have the freedom of choosing the function λ.

4.4.4 Solution of the transport equation

It remains to find the first term V_0 of the WKB approximation (4.4.5). Using (4.4.9) we have

$$\partial_s V_0 = \nabla V_0\cdot\partial_s\boldsymbol{X} = \lambda\nabla V_0\cdot\nabla\theta.$$

Consequently we can also rewrite the transport equation (4.4.8) as

$$2\partial_s V_0 + \lambda\left(\nabla^2\theta\right)V_0 = 0. \tag{4.4.12}$$

Using the identity

$$\partial_s\left(\frac{J}{\lambda}\right) = J\nabla^2\theta, \tag{4.4.13}$$

where $J = \left|\dfrac{\partial(x,y,z)}{\partial(s,\alpha,\beta)}\right|$ is the Jacobian of the transformation $\boldsymbol{x} = \boldsymbol{X}(s,\alpha,\beta)$, we can rewrite (4.4.12) as

$$\begin{aligned}
2J\partial_s V_0 + \lambda\partial_s\left(\frac{J}{\lambda}\right)V_0 &= 0\\
J\partial_s\left(V_0^2\right) + \lambda V_0^2\partial_s\left(\frac{J}{\lambda}\right) &= 0\\
\left(\frac{J}{\lambda}\right)\partial_s\left(V_0^2\right) + V_0^2\partial_s\left(\frac{J}{\lambda}\right) &= 0\\
\partial_s\left(\frac{1}{\lambda}JV_0^2\right) &= 0,
\end{aligned}$$

and its general solution is

$$V_0(\boldsymbol{x}) = a_0\sqrt{\frac{\lambda(\boldsymbol{x})}{J(\boldsymbol{x})}}. \tag{4.4.14}$$

Imposing the boundary condition $V_0(\boldsymbol{x}_0) = f(\boldsymbol{x}_0)$, we obtain

$$V_0(\boldsymbol{x}) = f(\boldsymbol{x}_0)\sqrt{\frac{\lambda(\boldsymbol{x})J(\boldsymbol{x}_0)}{\lambda(\boldsymbol{x}_0)J(\boldsymbol{x})}}. \tag{4.4.15}$$

This is true provided $\theta(0,\alpha,\beta) = 0$ in (4.4.11) since otherwise we will get an additional exponential term from the WKB ansatz (4.4.5)

$$e^{i\omega\theta(\boldsymbol{x}_0)} = e^{i\omega\theta(0,\alpha,\beta)}.$$

We now prove the identity (4.4.13) in $\mathbb{R}^2$ but this easily extends to $\mathbb{R}^3$. The transformation in $\mathbb{R}^2$ is $\boldsymbol{x} = \boldsymbol{X}(s,\alpha)$ and its Jacobian is

$$J = \left|\frac{\partial(x,y)}{\partial(s,\alpha)}\right| = \partial_s x\partial_\alpha y - \partial_\alpha x\partial_s y.$$

Using chain rule and the ray equation (4.4.9) we obtain

$$\begin{aligned}
\partial_s J &= \partial_s\left(\partial_s x\right)\partial_\alpha y + \partial_s x\partial_s\left(\partial_\alpha y\right) - \partial_s\left(\partial_s y\right)\partial_\alpha x - \partial_s y\partial_s\left(\partial_\alpha x\right)\\
&= \partial_s\left(\partial_s x\right)\partial_\alpha y - \partial_\alpha\left(\partial_s x\right)\partial_s y + \partial_\alpha\left(\partial_s y\right)\partial_s x - \partial_s\left(\partial_s y\right)\partial_\alpha x\\
&= \partial_\alpha y\Big[\partial_s x\partial_s + \partial_s y\partial_y\Big]\left(\partial_s x\right) - \partial_s y\Big[\partial_\alpha x\partial_x + \partial_\alpha y\partial_y\Big]\left(\partial_s x\right)\\
&\quad + \partial_s x\Big[\partial_\alpha x\partial_x + \partial_\alpha y\partial_y\Big]\left(\partial_s y\right) - \partial_\alpha x\Big[\partial_s x\partial_x + \partial_s y\partial_y\Big]\left(\partial_s y\right)
\end{aligned}$$

$$
\begin{aligned}
&= \Big[\partial_\alpha y \partial_s x - \partial_s y \partial_\alpha x\Big] \partial_x\left(\partial_s x\right) + \Big[\partial_\alpha y \partial_s y \partial_y\left(\partial_s x\right) - \partial_s y \partial_\alpha y \partial_y\left(\partial_s x\right)\Big] \\
&\quad + \Big[\partial_s x \partial_\alpha y - \partial_\alpha x \partial_s y \partial_y\Big] \partial_y\left(\partial_s y\right) + \Big[\partial_s x \partial_\alpha x \partial_x\left(\partial_s y\right) - \partial_\alpha x \partial_s x \partial_x\left(\partial_s y\right)\Big] \\
&= J\partial_x\left(\partial_s x\right) + J\partial_y\left(\partial_s y\right) \\
&= J\nabla\cdot\left(\partial_s \boldsymbol{x}\right) \\
&= J\nabla\cdot\left(\lambda\nabla\theta\right).
\end{aligned}
$$

For any smooth function $q(\boldsymbol{x})$,

$$
\begin{aligned}
\partial_s\left(qJ\right) &= q\partial_s J + J\partial_s q \\
&= qJ\nabla\cdot\left(\lambda\nabla\theta\right) + J\nabla q\cdot\partial_s\boldsymbol{x} \\
&= J\Big[q\nabla\cdot\left(\lambda\nabla\theta\right)\Big] + J\Big[\nabla q\cdot\left(\lambda\nabla\theta\right)\Big] \\
&= J\nabla\cdot\left(q\lambda\nabla\theta\right).
\end{aligned}
$$

The identity (4.4.13) follows by choosing $q = 1/\lambda$.

4.4.5 Ray equation

We may now focus on solving the ray equation (4.4.9). To remove the θ dependence, let $\boldsymbol{X} = (X_1, X_2, X_3)$. Dividing (4.4.9) by λ and differentiating the resulting equation componentwise yields

$$
\begin{aligned}
\frac{\partial}{\partial s}\left[\frac{1}{\lambda}\frac{\partial X_i}{\partial s}\right] = \frac{\partial}{\partial s}\left(\frac{\partial\theta(\boldsymbol{x})}{\partial x_i}\right) &= \sum_{j=1}^{3}\frac{\partial x_i}{\partial s}\frac{\partial}{\partial x_j}\left(\frac{\partial\theta(\boldsymbol{x})}{\partial x_i}\right) \\
&= \left(\frac{\partial}{\partial x_i}\nabla\theta\right)\cdot\left(\frac{\partial\boldsymbol{X}}{\partial s}\right) \\
&= \left(\partial_{x_i}\nabla\theta\right)\cdot\left(\lambda\nabla\theta\right) \\
&= \frac{1}{2}\lambda\partial_{x_i}\left(\nabla\theta\cdot\nabla\theta\right) \\
&= \frac{1}{2}\lambda\partial_{x_i}\mu^2.
\end{aligned}
$$

In vector form, this equals

$$
\frac{\partial}{\partial s}\left(\frac{1}{\lambda}\frac{\partial}{\partial s}\boldsymbol{X}\right) = \lambda\mu\nabla\mu. \tag{4.4.16}
$$

We require two boundary conditions as (4.4.16) is a second-order equation in s. Recall that each ray starts on the initial surface S. Given any point $\boldsymbol{x}_0 \in S$, its ray satisfies

$$
\boldsymbol{X}\big|_{s=0} = \boldsymbol{x}_0. \tag{4.4.17}
$$

The second boundary condition is typically

$$
\left.\frac{\partial\boldsymbol{X}}{\partial s}\right|_{s=0} = \lambda_0\mu_0\boldsymbol{n}_0, \tag{4.4.18}
$$

where $\boldsymbol{n}_0$ is the unit outward normal at $\boldsymbol{x}_0$, $\lambda_0 = \lambda(0,\alpha,\beta)$ and $\mu_0 = \mu(0,\alpha,\beta)$.

We can also rewrite the ray equation (4.4.9) by taking the dot product of (4.4.9) against $\partial_s \boldsymbol{X}$:

$$\frac{\partial \boldsymbol{X}}{\partial s} \cdot \frac{\partial \boldsymbol{X}}{\partial s} = \lambda^2 \nabla\theta \cdot \nabla\theta = \lambda^2\mu^2.$$

If ℓ be the arc length along a ray, then

$$\ell = \int_0^s \|\partial_s \boldsymbol{X}\|\, ds = \int_0^s \lambda\mu\, ds.$$

Hence, s equals the arc length along a ray if we choose $\lambda\mu = 1$. Another common choice is $\lambda = 1$.

4.4.6 Summary for $\lambda = 1/\mu$

From (4.4.16), choosing $\lambda\mu = 1$ amounts to solving

$$\frac{\partial}{\partial s}\left(\mu \frac{\partial}{\partial s}\boldsymbol{X}\right) = \nabla\mu(\boldsymbol{X}) \tag{4.4.19a}$$

$$\boldsymbol{X}|_{s=0} = \boldsymbol{x}_0 \in S, \quad \partial_s \boldsymbol{X}|_{s=0} = \boldsymbol{n}_0. \tag{4.4.19b}$$

Once this is solved, the phase function becomes

$$\theta(\boldsymbol{X}) = \int_0^s \mu(\boldsymbol{X})\, d\sigma \tag{4.4.20}$$

and the amplitude is

$$V_0(\boldsymbol{x}) = f(\boldsymbol{x}_0)\sqrt{\frac{\mu(\boldsymbol{x}_0)J(\boldsymbol{x}_0)}{\mu(\boldsymbol{x})J(\boldsymbol{x})}}. \tag{4.4.21}$$

Finally, the WKB approximation for the outward propagating wave is

$$u(\boldsymbol{x}, t) \sim f(\boldsymbol{x}_0)\sqrt{\frac{\mu(\boldsymbol{x}_0)J(\boldsymbol{x}_0)}{\mu(\boldsymbol{x})J(\boldsymbol{x})}} \exp\left[i\omega\left(-t + \int_0^s \mu(\boldsymbol{X}(\sigma))\, d\sigma\right)\right], \tag{4.4.22}$$

where s is the value for which the solution of (4.4.19) satisfies $\boldsymbol{X}(s) = \boldsymbol{x}$.

Example 4.4.2. For constant μ, the ray equation (4.4.19) becomes

$$\frac{\partial^2 \boldsymbol{X}}{\partial s^2} = 0 \implies \boldsymbol{X}(s) = \boldsymbol{x}_0 + s\boldsymbol{n}_0.$$

The phase function is

$$\theta = \mu_0 \int_0^s d\sigma = \mu_0 s.$$

Thus, given a point $\boldsymbol{x}$ on the ray, $s = \boldsymbol{n}_0 \cdot (\boldsymbol{x} - \boldsymbol{x}_0)$ the WKB approximation is

$$u(\boldsymbol{x}, t) \sim f(\boldsymbol{x}_0)\sqrt{\frac{J(\boldsymbol{x}_0)}{J(\boldsymbol{x})}} \exp\left[i\left(\boldsymbol{k}\cdot(\boldsymbol{x} - \boldsymbol{x}_0) - \omega t\right)\right],$$

where $\boldsymbol{k} = \mu_0\omega\boldsymbol{n}_0$ is the wave vector for the ray. In $\mathbb{R}^2$, when the boundary surface is the circle of radius a, $\boldsymbol{n}_0$ is simply the position vector $\boldsymbol{x} - \boldsymbol{x}_0$ and s is then the distance from the circle. In polar coordinates (ρ, φ), the Jacobian is just ρ and

$$u(\boldsymbol{x}, t) \sim f(\boldsymbol{x}_0)\sqrt{\frac{a}{\rho}}e^{i\omega(\mu_0(\rho - a) - t)}.$$

4.4.7 Breakdown of the WKB solution

It is important to consider circumstances in which the solution (4.4.22) can go wrong:

1. It does not hold at turning points $\boldsymbol{x}$ of μ, *i.e.* $\mu(\boldsymbol{x}) = 0$. Nonetheless, this can be handled analogously to the one-dimensional case in Section 4.2 using boundary layer method.

2. A more likely complication arises when $J = 0$. Points where this occurs are called *caustics* and these arise when two or more rays intersect, which results in the breakdown of the characteristic coordinates (s, α, β). If a ray passes through a caustic, one picks up an additional factor in the WKB solution (4.4.22) of the form $e^{im\pi/2}$, where the integer m depends on the rank of the Jacobian matrix at the caustic.

3. A less obvious breakdown occurs when $\boldsymbol{X}(s) = \boldsymbol{x}$ has no solution. This happens with *shadow regions* and it is resolved by introducing the idea of *ray splitting*.

4.5 Problems

1. Use the WKB method to find an approximation of the following problem on $x \in [0, 1]$:

$$\varepsilon y'' + 2y' + 2y = 0, \quad y(0) = 0, \quad y(1) = 1.$$

Solution: We make a WKB ansatz of the form

$$y(x) \sim e^{\theta(x)/\varepsilon^\alpha} \left(y_0(x) + \varepsilon^\alpha y_1(x) + \dots\right). \tag{4.5.1}$$

Substituting (4.5.1) into the given differential equation yields

$$\begin{aligned} &\varepsilon \left[\varepsilon^{-2\alpha}\theta_x^2 y_0 + \varepsilon^{-\alpha}\left(\theta_{xx}y_0 + 2\theta_x y_0' + \theta_x^2 y_1\right) + \dots\right] \\ &+ 2\left[\varepsilon^{-\alpha}\theta_x y_0 + y_0' + \theta_x y_1 + \dots\right] + 2\left[y_0 + \varepsilon^\alpha y_1 + \dots\right] = 0. \end{aligned}$$

Balancing leading order terms of the first two terms we obtain $\alpha = 1$ and the $\mathcal{O}(1/\varepsilon)$ equation is the *eikonal equation*

$$\theta_x^2 + 2\theta_x = 0 = \theta_x\left(\theta_x + 2\right)$$

which has two general solutions:

$$\theta(x) \equiv c_1 \quad \text{or} \quad \theta(x) = -2x + c_2,$$

where c_1, c_2 are arbitrary constants. The $\mathcal{O}(1)$ equation, after simplifying using the eikonal equation, is the following:

$$\theta_{xx}y_0 + 2\theta_x y_0' + 2y_0' + 2y_0 = 0. \tag{4.5.2}$$

Suppose $\theta_x = 0$, then (4.5.2) reduces to $2y_0' + 2y_0 = 0$ and its general solution is

$$y_0(x) = a_0 e^{-x}.$$

Suppose $\theta_x = -2$, then (4.5.2) reduces to $-2y_0' + 2y_0 = 0$ and its general solution is

$$y_0(x) = b_0 e^x.$$

Thus a first-term approximation of the general solution of the original problem is

$$y \sim a_0 e^{-x} + b_0 e^x e^{-2x/\varepsilon} \sim a_0 e^{-x} + b_0 e^{x-2x/\varepsilon},$$

where we absorb the constants c_1, c_2 into a_0, b_0 respectively. Imposing the boundary conditions $y_0(0) = 0$ and $y_0(1) = 1$ results in two linear equations in terms of a_0 and b_0:

$$\begin{cases} a_0 + b_0 & = 0 \\ a_0 e^{-1} + b_0 e^{1-2/\varepsilon} & = 1 \end{cases} \implies a_0 = -b_0, \quad b_0 = \frac{e}{e^{2-2/\varepsilon} - 1}$$

Hence, a first-term WKB approximation is

$$\begin{aligned} y &\sim b_0 \left(-e^{-x} + e^{x-2x/\varepsilon}\right) \\ &\sim -b_0 \left(e^{-x} - e^{x-2x/\varepsilon}\right) \\ &\sim \frac{e}{1 - e^{2-2/\varepsilon}} \left(e^{-x} - e^{x-2x/\varepsilon}\right) \\ &\sim \frac{1}{1 - e^{2-2\varepsilon}} \left(e^{1-x} - e^{x+1-2x/\varepsilon}\right). \end{aligned}$$

2. Consider seismic waves propagating through the upper mantle of the Earth from a source on the Earth's surface. We want to use a WKB approximation in $\mathbb{R}^3$ to solve the equation

$$\nabla^2 v + \omega^2 \mu^2(r) v = 0,$$

where μ has spherical symmetry. Take $\lambda = 1/\mu$.

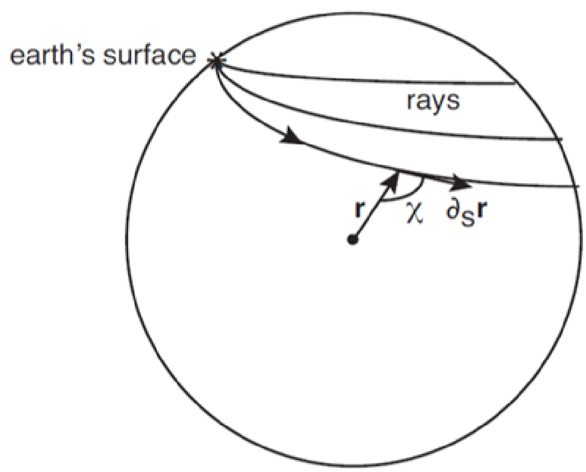


Figure 4.5: Rays representing waves propagating inside the earth from a source on the surface of the earth.

(a) Use the ray equation to show that the vector $\boldsymbol{p} = \boldsymbol{r} \times (\mu \partial_s \boldsymbol{r})$ is independent of s. Hence, show that $\|\boldsymbol{r}\|\mu \sin(\chi)$ is constant along a ray, where χ is the angle between $\boldsymbol{r}$ and $\partial_s \boldsymbol{r}$.

Solution: With the choice $\lambda = 1/\mu$, the ray equation (4.4.16) reduces to

$$\frac{\partial}{\partial s}\left(\mu \frac{\partial}{\partial s}\boldsymbol{X}\right) = \nabla\mu(\boldsymbol{X}), \quad \text{with } \boldsymbol{x} = \boldsymbol{r} = \boldsymbol{X}(s, \alpha, \beta).$$

Using the product rule for differentiating cross product we obtain

$$\begin{aligned}\partial_s \boldsymbol{p} &= \frac{\partial}{\partial s}\left(\boldsymbol{r} \times \mu \frac{\partial}{\partial s}\boldsymbol{r}\right) \\ &= \frac{\partial}{\partial s}\boldsymbol{r} \times \mu \frac{\partial}{\partial s}\boldsymbol{r} + \boldsymbol{r} \times \frac{\partial}{\partial s}\left(\mu \frac{\partial}{\partial s}\boldsymbol{r}\right) \\ &= \mu\left(\frac{\partial}{\partial s}\boldsymbol{r} \times \frac{\partial}{\partial s}\boldsymbol{r}\right) + \boldsymbol{r} \times \nabla\mu(\boldsymbol{r}).\end{aligned}$$

The first term vanishes because the cross product of any vector with itself is zero and the second term vanishes since $\boldsymbol{r}$ and $\nabla\mu(\boldsymbol{r})$ are parallel. Therefore $\partial_s \boldsymbol{p} = \boldsymbol{0}$ and so $\boldsymbol{p}$ is independent of s.

An immediate consequence of the previous result is that the vector $\boldsymbol{p}$ is constant along a ray, *i.e.* $\boldsymbol{p}$ has constant magnitude $\kappa > 0$ along a ray. First, the geometrical interpretation of the cross product gives the following

$$\|\boldsymbol{r}\|\|\partial_s \boldsymbol{r}\| \sin(\chi) = \|\boldsymbol{r} \times \partial_s \boldsymbol{r}\|,$$

where χ is the angle between the vectors $\boldsymbol{r}$ and $\partial_s \boldsymbol{r}$. Multiplying each side by the positive scalar function μ we obtain

$$\|\boldsymbol{r}\|\mu\|\partial_s \boldsymbol{r}\| \sin(\chi) = \|\boldsymbol{r} \times \mu \partial_s \boldsymbol{r}\| = \|\boldsymbol{p}\| = \kappa.$$

Using the ray equation and the eikonal equation,

$$\|\partial_s \boldsymbol{r}\|^2 = \partial_s \boldsymbol{r} \partial_s \boldsymbol{r} = (\lambda\nabla\theta) \cdot (\lambda\nabla\theta) = \lambda^2\mu^2 = 1,$$

since we take $\lambda = 1/\mu$. Hence,

$$\kappa = \|\boldsymbol{r}\|\mu\|\partial_s \boldsymbol{r}\| \sin(\chi) = \|\boldsymbol{r}\|\mu \sin(\chi) \quad \text{along a ray.} \tag{4.5.3}$$

(b) Part (a) implies that each ray lies in a plane containing the origin of the sphere. Let (ρ, φ) be polar coordinates of this plane. It follows that for a polar curve $\rho = \rho(\varphi)$, the angle χ satisfies

$$\sin(\chi) = \frac{\rho}{\sqrt{\rho^2 + (\partial_\varphi \rho)^2}}. \tag{4.5.4}$$

Assuming $\partial_\varphi \rho \neq 0$, show that

$$\varphi = \varphi_0 + \kappa \int_{\rho_0}^{\rho} \frac{dr}{r\sqrt{\mu^2 r^2 - \kappa^2}},$$

where $\rho_0, \varphi_0, \kappa$ are constants.

Solution: Given a ray, let (ρ, φ) be polar coodinates of the plane containing such ray. Since this plane contains the origin of the sphere, we can identify ρ as the magnitude of the radial (position) vector $\boldsymbol{r}$ and from (4.5.3) we know that

$$\sin(\chi) = \frac{\kappa}{\|\boldsymbol{r}\|\mu} = \frac{\kappa}{\mu\rho}. \tag{4.5.5}$$

Substituting (4.5.5) into (4.5.4) and rearranging we obtain

$$\begin{aligned}
\frac{\kappa}{\rho\mu} &= \frac{\rho}{\sqrt{\rho^2 + (\partial_\varphi \rho)^2}} \\
\sqrt{\rho^2 + (\partial_\varphi \rho)^2} &= \frac{\rho^2 \mu}{\kappa} \\
\rho^2 + (\partial_\varphi \rho)^2 &= \frac{\rho^4 \mu^2}{\kappa^2} \\
(\partial_\varphi \rho)^2 &= \frac{\rho^4 \mu^2}{\kappa^2} - \rho^2 \\
(\partial_\varphi \rho)^2 &= \frac{\rho^2}{\kappa^2}\left(\rho^2 \mu^2 - \kappa^2\right) \\
\partial_\varphi \rho &= \pm \frac{\rho}{\kappa}\sqrt{\rho^2 \mu^2 - \kappa^2}.
\end{aligned}$$

Assuming $\partial_\varphi \rho \neq 0$, we can invert this to obtain $\partial_\rho \varphi$. Therefore,

$$\begin{aligned}
\partial_\rho \varphi &= \pm \frac{\kappa}{\rho\sqrt{\rho^2 \mu^2 - \kappa^2}} \\
\varphi &= \varphi_0 \pm \kappa \int_{\rho_0}^{\rho} \frac{dr}{r\sqrt{\mu^2 r^2 - \kappa^2}},
\end{aligned}$$

where (φ_0, ρ_0) satisfies $\kappa = \pm \rho_0 \mu(\rho_0) \sin(\varphi_0)$.

(c) Use the definition of arc length, show that for a polar curve

$$\mu ds = \sqrt{\rho^2 + (\partial_\varphi \rho)^2}\, d\varphi. \tag{4.5.6}$$

Combining this result with part (b), show that the solution of the eikonal equation is given by

$$\theta = \frac{1}{\kappa} \int_{\varphi_0}^{\varphi} \mu^2 \rho^2 \, d\varphi.$$

Solution: First of all, we must distinguish the ray parameter s with the arclength parameter ℓ of a ray. For a polar curve $(\varphi, \rho(\varphi))$, we have $x = \rho(\varphi)\cos\varphi$ and $y = \rho(\varphi)\sin\varphi$ and so

$$\begin{aligned} d\ell &= \sqrt{\left(\frac{dx}{d\varphi}\right)^2 + \left(\frac{dy}{d\varphi}\right)^2} d\varphi \\ &= \sqrt{(-\rho\sin\varphi + \partial_\varphi\rho\cos\varphi)^2 + (\rho\cos\varphi + \partial_\varphi\rho\sin\varphi)^2} d\varphi \\ &= \sqrt{\rho^2\left(\sin^2\varphi + \cos^2\varphi\right) + (\partial_\varphi\rho)^2\left(\cos^2\varphi + \sin^2\varphi\right)} d\varphi \\ &= \sqrt{\rho^2 + (\partial_\varphi\rho)^2} d\varphi. \end{aligned}$$

Recall that the arclength ℓ along a ray satisfies

$$\ell = \int_0^s \lambda\mu\, ds.$$

It follows from the choice of $\lambda = 1/\mu$ that

$$d\ell = ds = \sqrt{\rho^2 + (\partial_\varphi\rho)^2}.$$

Since we take $\lambda = 1/\mu$, the solution of the eikonal equation is

$$\begin{aligned} \theta = \int_0^s \lambda\mu^2\, ds &= \int_0^s \mu\, ds \\ &= \int_{\varphi_0}^{\varphi} \mu\sqrt{\rho^2 + (\partial_\varphi\rho)^2}\, d\varphi \\ &= \int_{\varphi_0}^{\varphi} \frac{\mu\rho}{\sin(\chi)}\, d\varphi && \left[\text{From (4.5.4).}\right] \\ &= \int_{\varphi_0}^{\varphi} \mu\rho\left(\frac{\mu\rho}{\kappa}\right) d\varphi && \left[\text{From (4.5.5).}\right] \\ &= \frac{1}{\kappa}\int_{\varphi_0}^{\varphi} \mu^2\rho^2\, d\varphi, \end{aligned}$$

as desired. *Note: I did a dimensional analysis on the original expression* (4.5.6) *given in the problem and found out that μ is dimensionless, which is clearly false.*

Chapter 5

Method of Homogenization

5.1 Introductory Example

Consider the boundary value problem

$$\frac{d}{dx}\left(D\frac{du}{dx}\right) = f(x), \quad 0 < x < 1, \tag{5.1.1}$$

with $u(0) = a$ and $u(1) = b$. In many physical problems, D is known as the **conductivity tensor** and we are interested in $D = D(x, x/\varepsilon)$, where it includes a slow variation in x as well as a fast variation over a length scale that is $\mathcal{O}(\varepsilon)$. A physical realisation of this is a material having micro and macrostructures with spatial variation. For example, we might have

$$D(x, y) = \frac{1}{1 + \alpha x + \beta g(x)\cos y}, \tag{5.1.2}$$

with

$$\alpha = 0.1, \quad \beta = 0.1, \quad \varepsilon = 0.01, \quad g(x) = e^{4x(x-1)}.$$

Our main goal is to try to replace, if possible, $D(x, x/\varepsilon) = D(x, y)$ with some effective (averaged) D that is independent of ε. A naive guess would be to simply average over the fast variation, *i.e.*

$$\langle D\rangle_\infty = \lim_{y\to\infty}\frac{1}{y}\int_0^y D(x, r)\, dr. \tag{5.1.3}$$

For the given example (5.1.2), we have that

$$\langle D\rangle_\infty = \left[(1 + \alpha x)^2 - (\beta g(x))^2\right]^{-1/2}. \tag{5.1.4}$$

It turns out that this is not a good approximation because the solution of (5.1.1) with $\langle D\rangle_\infty$ might be a bad approximation of the solution of (5.1.1).

Because of the two different length scales in (5.1.1), it is natural to invoke the method of multiple scales, but with an important distinction. Here, we want to eliminate the fast length scale $y = x/\varepsilon$, as opposed to the standard multiple scales where we keep both the slow and normal scales. For the existence of solution of (5.1.1), we assume $D(x, y)$ is smooth and satisfies

$$0 < D_m(x) \le D(x, y) \le D_M(x) \tag{5.1.5}$$

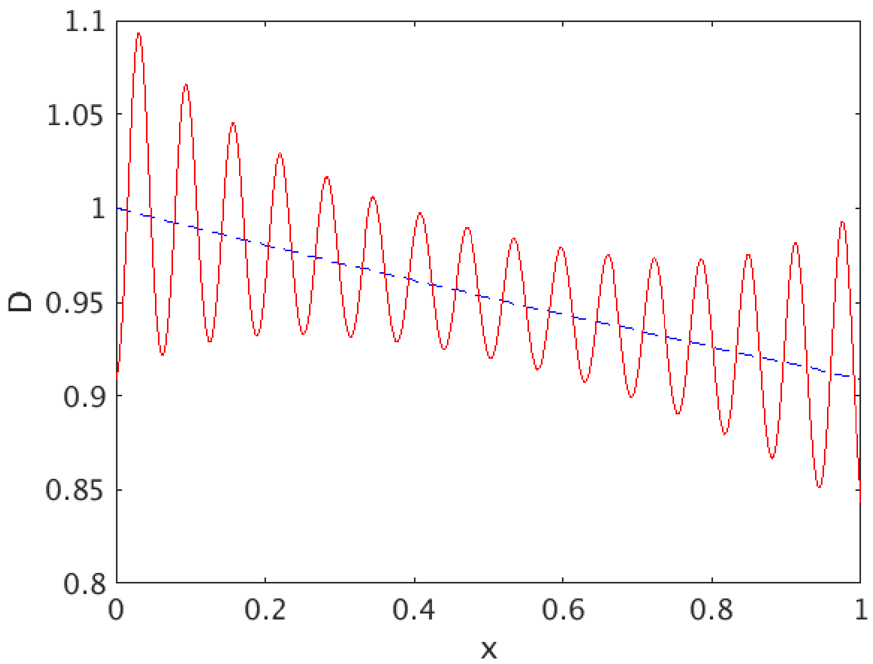


Figure 5.1: Rapidly varying coefficient D and its average. The red line depicts rapidly varying $D(x, x/\epsilon)$ in (5.1.2) and the blue dotted line shows its effective mean $\overline{D}(x) = 1/(1+\alpha x)$.

for all $x \in [0,1]$ and $y > 0$, where $D_m(x)$ and $D_M(x)$ are both continuous. With the fast scale $y = x/\varepsilon$ and the slow scale x, the derivative becomes

$$\frac{d}{dx} \longrightarrow \frac{1}{\varepsilon}\partial_y + \partial_x$$

and (5.1.1) becomes

$$(\partial_y + \varepsilon\partial_x)[D(x,y)(\partial_y + \varepsilon\partial_x)u] = \varepsilon^2 f(x). \tag{5.1.6}$$

We assume a regular perturbation expansion of the form

$$u \sim u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \ldots,$$

with $u_0, u_1, u_2, \ldots$ smooth, bounded functions of y. The $\mathcal{O}(1)$ equation is

$$\partial_y[D(x,y)\partial_y u_0] = 0,$$

and its general solution is

$$u_0(x,y) = c_1(x) + c_0(x)\int_{y_0}^{y} \frac{ds}{D(x,s)}, \tag{5.1.7}$$

where y_0 is some fixed but arbitrary number. In order fo u_0 to be bounded, we require $c_0 = 0$, since the associated integral in (5.1.7) is unbounded. Indeed, from the assumption (5.1.5), if $y > y_0$, then

$$\int_{y_0}^{y} \frac{ds}{D_M(x)} \le \int_{y_0}^{y} \frac{ds}{D(x,s)},$$

and it follows that

$$\frac{y-y_0}{D_M(x)} \leq \int_{y_0}^{y} \frac{ds}{D(x,s)}.$$

Since the left-hand side becomes infinite as $y \longrightarrow \infty$, so does the right-hand side. Therefore, $u_0 = u_0(x) = c_1(x)$. At this point, it is worth noting that

$$\frac{y-y_0}{D_M(x)} \leq \int_{y_0}^{y} \frac{ds}{D(x,s)} \leq \frac{y-y_0}{D_m(x)}, \tag{5.1.8}$$

i.e. the integral is unbounded but its growth is confined by linear functions in y as $y \longrightarrow \infty$.

The $\mathcal{O}(\varepsilon)$ equation is

$$\partial_y[D(x,y)\partial_y u_1] = -\partial_x u_0 \cdot \partial_y D. \tag{5.1.9}$$

Integrating this with respect to y twice and using the fact that $u_0 = u_0(x)$ yields

$$\begin{aligned} D(x,y)\partial_y u_1 &= b_0(x) - \partial_x u_0 D(x,y) \\ \partial_y u_1 &= \frac{b_0(x)}{D(x,y)} - \partial_x u_0 \\ u_1(x,y) &= \underbrace{b_1(x)}_{①} + \underbrace{b_0(x)\int_{y_0}^{y}\frac{ds}{D(x,s)}}_{②} - \underbrace{y\partial_x u_0}_{③}. \end{aligned} \tag{5.1.10}$$

Observe that ② and ③ increases linearly with y for large y, and analogous to removing secular terms in multiple scales, we require that these two terms cancel each other so that u_1 is bounded. This means that we must impose

$$\lim_{y\to\infty} \frac{1}{y}\left[b_0(x)\int_{y_0}^{y}\frac{ds}{D(x,s)} - y\partial_x u_0\right] = 0.$$

This can be rewritten as

$$\partial_x u_0(x) = \langle D^{-1}\rangle_\infty b_0(x), \quad \text{where} \quad \langle D^{-1}\rangle_\infty = \lim_{y\to\infty}\frac{1}{y}\int_{y_0}^{y}\frac{ds}{D(x,s)}. \tag{5.1.11}$$

In general multiple scales problem, it is enough to get information from $\mathcal{O}(\varepsilon)$ terms to obtain a first-term approximation. However, for homogenization problems, we need to proceed to $\mathcal{O}(\varepsilon^2)$ equation to determine $u_0(x)$. The $\mathcal{O}(\varepsilon^2)$ equation is

$$\partial_y[D(x,y)\partial_y u_2] = f(x) - b_0' - \partial_y[D(x,y)\partial_x u_1],$$

and integrating twice with respect to y gives the general solution

$$u_2(x,y) = d_1(x) + d_0(x)\int_{y_0}^{y}\frac{ds}{D(x,s)} - \int_{y_0}^{y}\partial_x u_1(x,s)\,ds + (f-b_0')\int_{y_0}^{y}\frac{s\,ds}{D(x,s)}. \tag{5.1.12}$$

The last integral is $\mathcal{O}(y^2)$ for large y and cannot be cancelled by other terms in (5.1.12). Therefore, we require $b_0'(x) = f(x)$. Finally, rearranging (5.1.11) and differentiating with respect to x we obtain

$$\partial_x[\overline{D}(x)\partial_x u_0(x)] = b_0'(x) = f(x), \tag{5.1.13}$$

where $\overline{D}(x)$ is the **harmonic mean** of D, defined as

$$\overline{D}(x) = \langle D^{-1} \rangle_\infty^{-1} = \lim_{y \to \infty} \frac{y}{\int_{y_0}^{y} \frac{ds}{D(x,s)}}. \tag{5.1.14}$$

We called (5.1.13) the **homogenized differential equation** with the **homogenized**, or **effective**, coefficient $\overline{D}$.

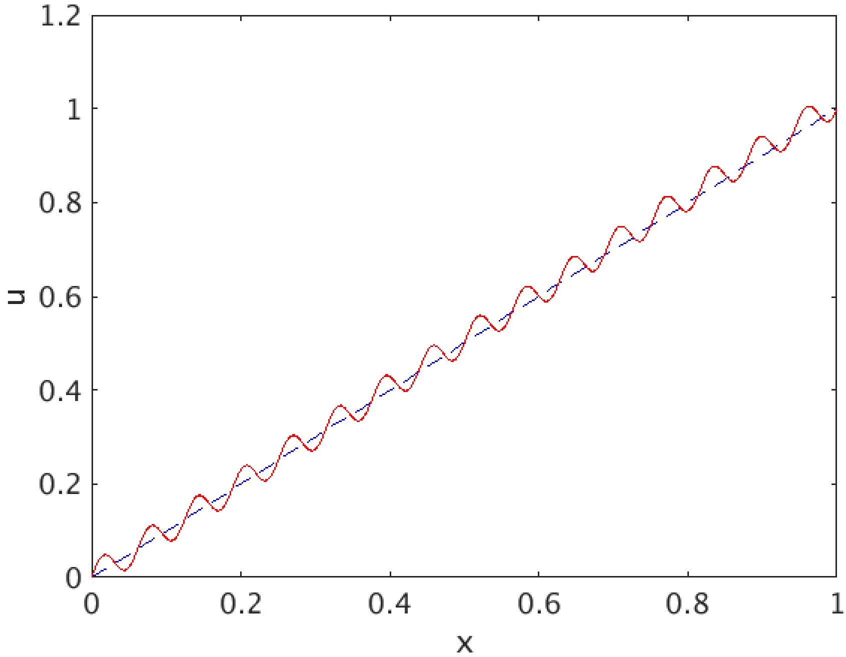


Figure 5.2: Exact and averaged solution. The red line depicts the exact solution and the blue line shows its homogenized solution.

Example 5.1.1. Given $D(x, y)$ in (5.1.2), it follows that

$$\langle D^{-1} \rangle_\infty = \lim_{y \to \infty} \frac{1}{y} \int_{y_0}^{y} \Big(1 + \alpha x + \beta g(x) \cos(s) \Big)\, ds = 1 + \alpha x. \tag{5.1.15}$$

In particular, $\overline{D} = 1$ for $\alpha = 0$. For $f(x) = 0$, $\alpha = 0$ and $\beta \neq 0$, the solution of (5.1.13), with $u_0(0) = 0$ and $u_0(1) = 1$, is $u_0(x) = x$. In Figure 5.2 we compare $u_0(x)$ with the exact solution of (5.1.1)

$$u(x) = \frac{x + \varepsilon\beta \sin(x/\varepsilon)}{1 + \varepsilon\beta \sin(1/\varepsilon)}.$$

5.2 Multi-dimensional Problem: Periodic Substructure

Given an open, connected, smooth region $\Omega \subset \mathbb{R}^n$, consider the inhomogeneous Dirichlet problem

$$\nabla \cdot (D \nabla u) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{5.2.1a}$$

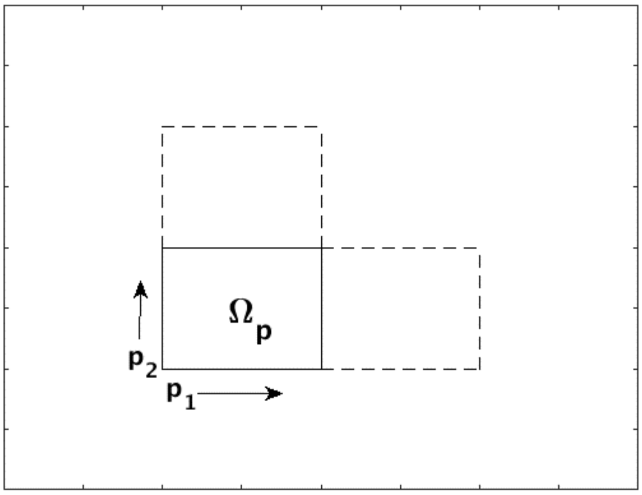


Figure 5.3: Fundamental domain with periodic substructure. On the fundamental domain, function has a same set of values.

$$u = g(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega. \tag{5.2.1b}$$

The coefficient $D = D(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)$ is assumed to be positive and smooth, and because (5.2.1) is harder to solve compared to (5.1.1), we also assume that D is periodic in the fast scale $\boldsymbol{y} = \boldsymbol{x}/\varepsilon$. In other words, there is a period vector $\boldsymbol{y}_p$ with positive entries such that

$$D(\boldsymbol{x}, \boldsymbol{y} + \boldsymbol{y}_p) = D(\boldsymbol{x}, \boldsymbol{y}) \quad \text{for all } \boldsymbol{x}, \boldsymbol{y}. \tag{5.2.2}$$

5.2.1 Periodicity of $D(\boldsymbol{x}, \boldsymbol{y})$

Suppose

$$D = D(\boldsymbol{y}) = y + \cos(2y_1 - 3y_2).$$

One finds that $\boldsymbol{y}_p = (\pi, 2\pi/3)$ and this means that we can determine D anywhere in $\mathbb{R}^2$ if we know its values in the rectangle $(y_1, y_2) \in [\alpha_0, \alpha_0 + \pi] \times [\beta_0, \beta_0 + 2\pi/3]$ for arbitrary $\alpha_0, \beta_0 \in \mathbb{R}$. This structure motivates the definition of a *cell* (or *fundamental domain*), Ω_p. Mathematically, given $\boldsymbol{y}_p = (p_1, p_2)$, Ω_p is the rectangle

$$\Omega_p = [\alpha_0, \alpha_0 + p_1] \times [\beta_0, \beta_0 + p_2],$$

where α_0, β_0 are given arbitrary constants that must be consistent with Ω. It is possible for the period vector $\boldsymbol{y}_p$ to depend on the slow variable $\boldsymbol{x}$. For example, consider

$$D(\boldsymbol{x}, \boldsymbol{y}) = 6 + \cos\left(y_1 e^{x_2} + 4y_2\right).$$

One finds that $\boldsymbol{y}_p = (2\pi e^{x_2}, \pi/2)$.

An important consequence of periodicity is values of a function on the boundary of the fundamental domain is also periodic. Suppose y_L and y_R are points on the left-hand and right-hand boundary of the fundamental domain respectively. For any C^2 periodic functions w, we have

$$\begin{cases} w(y_L) & = w(y_R) \\ \nabla_{\boldsymbol{y}} w(y_L) & = \nabla_{\boldsymbol{y}} w(y_R) \\ \partial_{y_i}\partial_{y_j} w(y_L) & = \partial_{y_i}\partial_{y_j} w(y_R) \end{cases} \tag{5.2.3}$$

These conditions must hold at upper and lower boundary as well.

5.2.2 Homogenization procedure

Setting $\boldsymbol{y} = \boldsymbol{x}/\varepsilon$, the derivative becomes

$$\nabla \longrightarrow \nabla_{\boldsymbol{x}} + \frac{1}{\varepsilon}\nabla_{\boldsymbol{y}}.$$

Substituting this into (5.2.1) and multiplying each side by ε^2 yields

$$(\nabla_{\boldsymbol{y}} + \varepsilon\nabla_{\boldsymbol{x}})\left[D(\boldsymbol{x},\boldsymbol{y})(\nabla_{\boldsymbol{y}} + \varepsilon\nabla_{\boldsymbol{x}})u(\boldsymbol{x},\boldsymbol{y})\right] = \varepsilon^2 f(\boldsymbol{x}). \tag{5.2.4}$$

We introduce an asymptotic expansion of the form

$$u \sim u_0(\boldsymbol{x},\boldsymbol{y}) + \varepsilon u_1(\boldsymbol{x},\boldsymbol{y}) + \varepsilon^2 u_2(\boldsymbol{x},\boldsymbol{y}) + \dots$$

and we assume that $u_0, u_1, u_2, \dots$ are periodic in $\boldsymbol{y}$ with period $\boldsymbol{y}_p$ due to the periodicity assumption on $\mathcal{D}$.

The $\mathcal{O}(1)$ equation is

$$\nabla_{\boldsymbol{y}}(D\nabla_{\boldsymbol{y}}u_0) = 0,$$

and the general solution of this, which is bounded, is $u_0 = u_0(\boldsymbol{x})$. If D were constant, then it follows from *Liouville's theorem* that bounded solutions of Laplace's equation over $\mathbb{R}^2$ are constants. One can argue similarly in the case where D is not constant. The $\mathcal{O}(\varepsilon)$ equation is

$$\nabla_{\boldsymbol{y}} \cdot (D\nabla_{\boldsymbol{y}}u_1) = -(\nabla_{\boldsymbol{y}}D)\cdot(\nabla_{\boldsymbol{x}}u_0). \tag{5.2.5}$$

Because u_1 is periodic in $\boldsymbol{y}$, it suffices to solve (5.2.5) in a cell Ω_p and then simply extend the solution using periodicity. Observe that (5.2.5) is linear with respect to $\boldsymbol{y}$ and u_0 does not depend on $\boldsymbol{y}$. Thus the general solution of (5.2.5) follows from superposition principle

$$u_1(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{a}\cdot\nabla_{\boldsymbol{x}}u_0 + c(\boldsymbol{x}), \tag{5.2.6}$$

with $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\boldsymbol{y})$ periodic in $\boldsymbol{y}$, satisfying

$$\nabla_{\boldsymbol{y}}\cdot(D\nabla_{\boldsymbol{y}}\boldsymbol{a}_i) = -\partial_{y_i}D \quad \text{for } \boldsymbol{y}\in\Omega_p. \tag{5.2.7}$$

The $\mathcal{O}(\varepsilon^2)$ eqution is

$$\nabla_{\boldsymbol{y}}\cdot[D(\nabla_{\boldsymbol{y}}u_2 + \nabla_{\boldsymbol{x}}u_1)] + \nabla_{\boldsymbol{x}}\cdot[D(\nabla_{\boldsymbol{y}}u_1 + \nabla_{\boldsymbol{x}}u_0)] = f(\boldsymbol{x}). \tag{5.2.8}$$

To derive the homogenized equation for u_0, we introduce the cell average of a function $v(\boldsymbol{x},\boldsymbol{y})$ over Ω_p:

$$\langle v\rangle_p(\boldsymbol{x}) = \frac{1}{|\Omega_p|}\int_{\Omega_p} v(\boldsymbol{x},\boldsymbol{y})\,dV_{\boldsymbol{y}}.$$

Averaging the first term of (5.2.8) and applying the divergence theorem gives

$$\begin{aligned}
\Big\langle \nabla_{\boldsymbol{y}}\cdot[D(\nabla_{\boldsymbol{y}}u_2 + \nabla_{\boldsymbol{x}}u_1)]\Big\rangle_p &= \frac{1}{|\Omega_p|}\int_{\Omega_p}\nabla_{\boldsymbol{y}}\cdot[D(\nabla_{\boldsymbol{y}}u_2 + \nabla_{\boldsymbol{x}}u_1)]\,dV_{\boldsymbol{y}} \\
&= \frac{1}{|\Omega_p|}\int_{\partial\Omega_p} D\boldsymbol{n}\cdot(\nabla_{\boldsymbol{y}}u_2 + \nabla_{\boldsymbol{x}}u_1)\,dS_{\boldsymbol{y}} \\
&= 0
\end{aligned}$$

since u_1, u_2 are periodic over the cell Ω_p. Next, using (5.2.6) we have

$$\begin{aligned}\langle D\partial_{y_i} u_1\rangle_p &= \langle D\partial_{y_i}(\boldsymbol{a}\cdot\nabla_{\boldsymbol{x}} u_0)\rangle_p \\ &= \langle D\partial_{y_i}\boldsymbol{a}\rangle_p\cdot\nabla_{\boldsymbol{x}} u_0.\end{aligned}$$

Similarly,

$$\langle D\partial_{x_i} u_0\rangle_p = \langle D\rangle_p \partial_{x_i} u_0 \implies \Big\langle \nabla_{\boldsymbol{x}}\cdot(D\nabla_{\boldsymbol{x}} u_0)\Big\rangle_p = \nabla_{\boldsymbol{x}}\cdot\Big(\langle D\rangle_p\nabla_{\boldsymbol{x}} u_0\Big)$$

Combining everything, the average of (5.2.8) is

$$\nabla_{\boldsymbol{x}}\cdot\Big[\langle D\nabla_{\boldsymbol{y}}\boldsymbol{a}\rangle_p\cdot\nabla_{\boldsymbol{x}} u_0\Big] + \nabla_{\boldsymbol{x}}\cdot\Big[\langle D\rangle_p\nabla_{\boldsymbol{x}} u_0\Big] = f(\boldsymbol{x}).$$

We can rewrite the homogenized problem in a more compact fashion:

$$\nabla_{\boldsymbol{x}}\cdot\Big[\underline{\underline{D}}\nabla_{\boldsymbol{x}} u_0\Big] = f(\boldsymbol{x}) \quad \text{for } \boldsymbol{x}\in\Omega, \tag{5.2.9a}$$

$$u_0 = g(\boldsymbol{x}) \quad \text{for } x\in\partial\Omega, \tag{5.2.9b}$$

$$\underline{\underline{D}} = \langle D\nabla_{\boldsymbol{y}}\boldsymbol{a}\rangle_p + \langle D\rangle_p \underline{\underline{I}}. \tag{5.2.9c}$$

In $\mathbb{R}^2$, the homogenized coefficients are

$$\underline{\underline{D}} = \begin{bmatrix}\langle D\rangle_p + \langle D\partial_{y_1} a_1\rangle_p & \langle D\partial_{y_1} a_2\rangle_p \\ \langle D\partial_{y_2} a_1\rangle_p & \langle D\rangle_p + \langle D\partial_{y_2} a_2\rangle_p\end{bmatrix} \tag{5.2.10}$$

and the functions a_i are smooth periodic solutions of the cell problem

$$\nabla_{\boldsymbol{y}}\cdot(D\nabla_{\boldsymbol{y}} a_i) = -\partial_{y_i} D \quad \text{for } \boldsymbol{y}\in\Omega_p. \tag{5.2.11}$$

Example 5.2.1. Consider the cell $\Omega_p = [0,a]\times[0,b]$ in $\mathbb{R}^2$. To determined the homogenized coefficients in (5.2.10), it is necessary to solve the cell problem (5.2.11). Consider a "separable" coefficient function D:

$$D(\boldsymbol{x},\boldsymbol{y}) = D_0(x_1,x_2)e^{\alpha(y_1)}e^{\beta(y_2)},$$

where $\alpha(y_1)$ and $\beta(y_2)$ are periodic with period a and b respectively. The cell equations for a_1, a_2 are

$$\begin{aligned}\partial_{y_1}(D\partial_{y_1} a_1) + \partial_{y_2}(D\partial_{y_2} a_1) &= -\partial_{y_1} D \\ \partial_{y_1}(D\partial_{y_1} a_2) + \partial_{y_2}(D\partial_{y_2} a_2) &= -\partial_{y_2} D.\end{aligned}$$

Taking $a_1 = a_1(y_1)$ and $a_2 = a_2(y_2)$, it follows that

$$\begin{aligned}e^{\alpha(y_1)}\partial_{y_1} a_1 &= \kappa_1 - e^{\alpha(y_1)} \\ e^{\beta(y_2)}\partial_{y_2} a_2 &= \kappa_2 - e^{\beta(y_2)}\end{aligned}$$

and

$$a_1(y_1) = -y_1 + \kappa_1\int_0^{y_1} e^{-\alpha(s)}\,ds$$

$$a_2(y_2) = -y_2 + \kappa_2 \int_0^{y_2} e^{-\beta(s)}\, ds.$$

From the periodicity of a_1 and a_2, *i.e.*

$$a_1(0) = a_1(a), \quad a_2(0) = a_2(b),$$

one finds that

$$\kappa_1 = a\left(\int_0^a e^{-\alpha(s)}\, ds\right)^{-1}, \quad \kappa_2 = b\left(\int_0^b e^{-\beta(s)}\, ds\right)^{-1}.$$

Now, since $\partial_{y_2} a_1 = \partial_{y_1} a_2 = 0$, it follows from (5.2.10) that $D_{12} = D_{21} = 0$. Moreover,

$$\begin{aligned}
\langle D\partial_{y_1} a_1\rangle_p &= \frac{1}{ab}\int_0^a\int_0^b D_0(\boldsymbol{x})e^{\alpha(y_1)+\beta(y_2)}\Big(-1+\kappa_1 e^{-\alpha(y_1)}\Big)\, dy_1\, dy_2 \\
&= -\frac{1}{ab}\int_0^a\int_0^b D_0(\boldsymbol{x})e^{\alpha(y_1)+\beta(y_2)}\, dy_1\, dy_2 + \frac{1}{ab}\int_0^a\int_0^b D_0(\boldsymbol{x})\kappa_1 e^{\beta(y_2)}\, dy_1\, dy_2 \\
&= -\langle D\rangle_p + D_0(\boldsymbol{x})\kappa_1\left(\frac{1}{b}\int_0^b e^{\beta(s)}\, ds\right) \\
&= -\langle D\rangle_p + D_0(\boldsymbol{x})\left(\frac{\kappa_1}{\kappa_2}\right),
\end{aligned}$$

and similarly

$$\begin{aligned}
\langle D\partial_{y_2} a_2\rangle_p &= \frac{1}{ab}\int_0^a\int_0^b D_0(\boldsymbol{x})e^{\alpha(y_1)+\beta(y_2)}\Big(-1+\kappa_2 e^{-\beta(y_2)}\Big)\, dy_1\, dy_2 \\
&= -\frac{1}{ab}\int_0^a\int_0^b D_0(\boldsymbol{x})e^{\alpha(y_1)+\beta(y_2)}\, dy_1\, dy_2 + \frac{1}{ab}\int_0^a\int_0^b D_0(\boldsymbol{x})\kappa_2 e^{\alpha(y_1)}\, dy_1\, dy_2 \\
&= -\langle D\rangle_p + D_0(\boldsymbol{x})\kappa_2\left(\frac{1}{a}\int_0^a e^{\alpha(s)}\, ds\right) \\
&= -\langle D\rangle_p + D_0(\boldsymbol{x})\left(\frac{\kappa_2}{\kappa_1}\right).
\end{aligned}$$

Consequently, the homogenized differential equation (5.2.9) for u_0 is

$$\partial_{x_1}(D_1\partial_{x_1}u_0) + \partial_{x_2}(D_2\partial_{x_2}u_0) = 0,$$

where $D_i(\boldsymbol{x}) = \lambda_i D_0(\boldsymbol{x})$, with

$$\lambda_1 = \frac{\kappa_1}{\kappa_2}, \quad \lambda_2 = \frac{\kappa_2}{\kappa_1}.$$

Interestingly, for D_1 we get the harmonic mean of $e^{\alpha(y_1)}$ multiplied by the arithmetic mean of $e^{\beta(y_2)}$, and vice versa for D_2.

5.3 Problem

1. Consider the equation

$$\partial_x\left(D\partial_x u\right) + g(u) = f(x, x/\varepsilon), \quad 0 < x < 1, \tag{5.3.1}$$

with $u = 0$ when $x = 0, 1$. Assume $D = D(x, x/\varepsilon)$. Use the method of multiple-scales to show that the leading order homogenised equation is

$$\partial_x \left(\overline{D}\partial_x u_0\right) + g(u_0) = \langle f \rangle_\infty,$$

where $\overline{D}$ is the harmonic mean of D and

$$\langle f \rangle_\infty = \lim_{y\to\infty} \left(\frac{1}{y} \int_{y_0}^{y} f(x,s)\, ds \right).$$

We assume the coefficient $D(x, y)$ is smooth and satisfies

$$0 < D_m(x) \leq D(x,y) \leq D_M(x),$$

for some continuous functions D_m, D_M in $[0, 1]$. We introduce $y = x/\varepsilon$ and designate the slow scale simply as x. The derivative transforms into

$$\frac{d}{dx} \longrightarrow \frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial y} = \partial_x + \frac{1}{\varepsilon}\partial_y,$$

and (5.3.1) becomes

$$(\partial_y + \varepsilon\partial_x)\left[D(x,y)\left(\partial_y + \varepsilon\partial_x\right)u\right] + \varepsilon^2 g(u) = \varepsilon^2 f(x,y). \tag{5.3.2}$$

We take a regular asymptotic expansion

$$u \sim u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \dots, \tag{5.3.3}$$

where we assume that $u_n, n = 0, 1, \dots$ are bounded functions of y. We now substitute (5.3.3) into (5.3.2) and collect terms of same order.

The $\mathcal{O}(1)$ equation is

$$\partial_y\left[D(x,y)\partial_y u_0\right] = 0,$$

and its general solution is

$$u_0(x,y) = c_1(x) + c_0(x)\int_{y_0}^{y} \frac{ds}{D(x,s)},$$

with y_0 fixed. We deduce from the lecture that $c_0(x)$ must be zero and consequently u_0 is a function of x only, *i.e.* $u_0(x, y) = u_0(x)$.

The $\mathcal{O}(\varepsilon)$ equation is

$$\partial_y\left[D(x,y)\partial_y u_1\right] = -\partial_x u_0 \partial_y D,$$

and its general solution is

$$u_1(x,y) = b_1(x) + b_0(x)\int_{y_0}^{y} \frac{ds}{D(x,s)} - y\partial_x u_0.$$

We deduce from the lecture that the following equation must be true to prevent u_1 from blowing up:

$$\partial_x u_0 = \langle D^{-1}\rangle_\infty b_0(x), \tag{5.3.4}$$

where $\langle D^{-1}\rangle_\infty = (\overline{D})^{-1}$.

The $\mathcal{O}(\varepsilon^2)$ equation is

$$\partial_y\Big[D(x,y)\partial_y u_2\Big] = f(x,y) - \partial_x b_0 - g(u_0) - \partial_y\Big(D\partial_x u_1\Big),$$

and solving this yields

$$\begin{aligned}
D(x,y)\partial_y u_2 &= a_0(x) + \int_{y_0}^{y} f(x,s)\,ds - \Big[\partial_x b_0 + g(u_0)\Big]y - D\partial_x u_1\\
\partial_y u_2 &= \frac{a_0(x)}{D(x,y)} - \partial_x u_1 - \frac{\Big[\partial_x b_0 + g(u_0)\Big]y}{D(x,y)} + \frac{1}{D(x,y)}\int_{y_0}^{y} f(x,s)\,ds\\
u_2(x,y) &= d_1(x) + d_0(x)\int_{y_0}^{y}\frac{ds}{D(x,s)} - \int_{y_0}^{y}\partial_x u_1(x,s)\,ds\\
&\quad + \int_{y_0}^{y}\left\{\frac{1}{D(x,\tau)}\left(-\Big[\partial_x b_0 + g(u_0)\Big]\tau + \int_{y_0}^{\tau} f(x,s)\,ds\right)\right\}d\tau
\end{aligned}$$

Since the last integral is $\mathcal{O}(y^2)$ for large y and there are no other terms in the expression of $u_2(x,y)$ that can cancel this growth, it is necessary to impose

$$\lim_{y\to\infty}\frac{1}{y^2}\int_{y_0}^{y}\left\{\frac{1}{D(x,\tau)}\left(-\Big[\partial_x b_0 + g(u_0)\Big]\tau + \int_{y_0}^{\tau} f(x,s)\,ds\right)\right\}d\tau = 0,$$

A slightly weaker requirement is

$$\lim_{\tau\to\infty}\frac{1}{\tau}\left(\int_{y_0}^{\tau} f(x,s)\,ds - \Big[\partial_x b_0 + g(u_0)\Big]\tau\right) = 0,$$

or equivalently

$$\partial_x b_0 + g(u_0) = \lim_{\tau\to\infty}\frac{1}{\tau}\int_{y_0}^{\tau} f(x,s)\,ds = \langle f\rangle_\infty. \tag{5.3.5}$$

Differentiating (5.3.4) and using the relation (5.3.5), it follows that

$$\overline{D}\partial_x u_0 = b_0 \implies \partial_x\Big(\overline{D}\partial_x u_0\Big) = \partial_x b_0 = \langle f\rangle_\infty - g(u_0),$$

and the leading order homogenised equation follows.

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